

THE COMPLEX TIME SEGAL–BARGMANN TRANSFORM

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ABSTRACT. We introduce a new form of the generalized Segal–Bargmann transform for a Lie group K of compact type. We show that the heat kernel $(\rho_t(x))_{t>0, x \in K}$ has a space-time analytic continuation to a holomorphic function $(\rho_{\mathbb{C}}(\tau, z))_{\operatorname{Re} \tau > 0, z \in K_{\mathbb{C}}}$, where $K_{\mathbb{C}}$ is the complexification of the Lie group K .

Let $s > 0$ and $\tau \in \mathbb{C}$ be such that $\operatorname{Re} \tau > 0$ and $s > |\tau|^2/2\operatorname{Re} \tau$. For $f \in L^2(K, \rho_s)$, the new transform is defined by the integral

$$(B_{s,\tau}f)(z) = \int_K f(x) \rho_{\mathbb{C}}(\tau, x^{-1}z) dx.$$

We construct a heat kernel density $\mu_{s,\tau}$ on $K_{\mathbb{C}}$ such, for all s, τ in the given parameter range, $B_{s,\tau}$ is an isometric isomorphism from $L^2(K, \rho_s)$ onto the space of holomorphic functions in $L^2(K_{\mathbb{C}}, \mu_{s,\tau})$. When $\tau = t > 0$ and $s = t$, the transform $B_{t,t}$ coincides with the one introduced by the second author for compact groups and extended by the first author to groups of compact type. When $\tau = t > 0$ the transform $B_{s,t}$ coincides with the one introduced by the first two authors.

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1. INTRODUCTION

1.1. The Classical Segal–Bargmann Transform. This paper concerns a generalization of the Segal–Bargmann transform over compact type Lie groups, to allow the time parameter of the transform to be complex. We begin by briefly discussing the history of the transform. For $t > 0$ and $k \in \mathbb{N}$, let γ_t denote the variance t Gaussian density on \mathbb{R}^k : $\gamma_t(x) = (2\pi t)^{-k/2} \exp(-|x|^2/2t)$. This is the *heat kernel* on \mathbb{R}^k : the solution u of the heat equation $\partial_t u = \frac{1}{2} \Delta u$ with (sufficiently integrable) initial condition f is given by convolution with γ_t ,

$$u(t, x) = (\gamma_t * f)(x) = \int_{\mathbb{R}^k} \gamma_t(x - y) f(y) dy. \quad (1.1)$$

The function γ_t admits an explicit entire analytic continuation to \mathbb{C}^k , which we call $(\gamma_t)_{\mathbb{C}}$: it is simply the function

$$(\gamma_t)_{\mathbb{C}}(z) = (2\pi t)^{-k/2} \exp\left(-\frac{z \cdot z}{2t}\right) = (2\pi t)^{-k/2} \exp\left(\frac{1}{2t} \sum_{j=1}^k z_j^2\right).$$

This function is *not* the heat kernel on $\mathbb{C}^k \cong \mathbb{R}^{2k}$, which (with a variance rescaling) we refer to as μ_t :

$$\mu_t(z) = (\pi t)^{-k} \exp(-|z|^2/t).$$

The function μ_t is bounded and in $L^p(\mathbb{C}^k)$ for all p ; the analytic function $(\gamma_t)_{\mathbb{C}}$ is unbounded and not in $L^p(\mathbb{C}^k)$ for any p . Nevertheless, the convolution formula (1.1) may still be used as a mollifier: if $f \in L^1_{\text{loc}}(\mathbb{R}^k)$ and of sufficiently slow growth, then the integral

$$(S_t f)(z) \equiv \int_{\mathbb{R}^k} f(y) (\gamma_t)_{\mathbb{C}}(z - y) dy \quad (1.2)$$

converges and defines an entire holomorphic function on \mathbb{C}^k . The map $f \mapsto S_t f$ is the **Segal–Bargmann transform**, invented and explored by the eponymous authors of [1, 2, 27, 28]. The main theorem is that S_t is an isometric isomorphism from $L^2(\mathbb{R}^k, \gamma_t)$ onto $\mathcal{H}L^2(\mathbb{C}^k, \mu_t)$ — the reproducing kernel Hilbert space of holomorphic functions in $L^2(\mathbb{C}^k, \mu_t)$.

1.2. The Segal–Bargmann Transform for Lie Groups of Compact Type. In [16], the second author introduced an analog of the Segal–Bargmann transform on an arbitrary compact Lie group. Then in [4], the first author extended the results of [16] to a Lie group K of compact type (cf. Section 2.2), a class that includes both compact groups and \mathbb{R}^k . The idea of [16] and [4] is the same as in the \mathbb{R}^k case: the heat kernel ρ_t on K (cf. (1.4) and Theorem 2.30) has an entire analytic continuation $(\rho_t)_{\mathbb{C}}$ to the *complexification* $K_{\mathbb{C}}$ of K

(cf. Section 2.1). The transform B_t is nominally defined by the group convolution formula generalizing (1.2):

$$(B_t f)(g) = \int_K f(y) (\rho_t)_\mathbb{C}(y^{-1}g) dy. \quad (1.3)$$

The theorem is that B_t is an isometric isomorphism from $L^2(K, \rho_t)$ onto the holomorphic space $\mathcal{H}L^2(K_\mathbb{C}, \mu_t)$, where μ_t is the (time-rescaled) heat kernel on $K_\mathbb{C}$.

Later, in [8, 17], the authors made a further generalization related to the time parameter t . One can use a different time s to measure the functions f in the domain:

$$B_{s,t}: L^2(K, \rho_s) \rightarrow \mathcal{H}L^2(K_\mathbb{C}, \mu_{s,t})$$

is still an isometric isomorphism for an appropriate two-parameter heat kernel type density $\mu_{s,t}$, provided $s > \frac{t}{2}$. The definition of the transform itself is unchanged, still given by (1.3), but the domain and range are different than in the original framework of [16]. In the special case that $K = \mathbb{R}^k$, the two-parameter heat kernel density $\mu_{s,t}$ in the range is a Gaussian measure with different variances in the real and imaginary directions, cf. Remark 4.4.

1.3. The Complex-Time Segal–Bargmann Transform. The topic of the present paper is a new generalization that modifies the transform $B_{s,t}$ as well; in particular, we show that the time parameter t can also be extended into the complex plane, and there is still an isomorphism between real and holomorphic L^2 spaces of associated heat kernel measures. This generalization is natural and, in a certain sense, a completion of Segal–Bargmann transform theory, as explained below.

Let K be a compact type Lie group with Lie algebra \mathfrak{k} , and fix an $\text{Ad}(K)$ -invariant inner product $\langle \cdot, \cdot \rangle_\mathfrak{k}$ on \mathfrak{k} (cf. Section 2.2). This induces a left-invariant Riemannian metric on K , and an associated Laplace operator $\Delta_\mathfrak{k}$, which is bi-invariant, elliptic, and essentially self-adjoint in $L^2(K)$ with respect to any right Haar measure (see Section 2.6 for precise statements and proofs of these properties). The self-adjoint contraction semigroup $\{e^{\frac{t}{2}\Delta_\mathfrak{k}}\}_{t>0}$ is given by a group convolution kernel ρ_t , the **heat kernel**:

$$\left(e^{\frac{t}{2}\Delta_\mathfrak{k}} f\right)(x) = \int_K f(y) \rho_t(y^{-1}x) dy. \quad (1.4)$$

The key properties of the heat kernel ρ_t are described in Theorem 2.30, Remark 2.31, and Corollary 2.32.

Our first theorem is that the heat kernel can be complexified in both space and time.

Theorem 1.1. *Let K be a compact type Lie group, with a given $\text{Ad}(K)$ -invariant inner product on its Lie algebra \mathfrak{k} , and let $(\rho_t)_{t>0}$ be the associated heat kernel. Let \mathbb{C}_+ denote the right half-plane $\{\tau = t + iu: t > 0, u \in \mathbb{R}\}$. There is a unique holomorphic function*

$$\rho_\mathbb{C}: \mathbb{C}_+ \times K_\mathbb{C} \rightarrow \mathbb{C}$$

such that $\rho_\mathbb{C}(t, x) = \rho_t(x)$ for $t > 0$ and $x \in K \subset K_\mathbb{C}$.

Theorem 1.1 is proved in Section 4.

Following (1.2) and (1.3), we can now putatively define the **complex time Segal–Bargmann transform** over K as the integral transform

$$(B_{s,\tau} f)(z) = \int_K f(x) \rho_\mathbb{C}(\tau, x^{-1}z) dx, \quad \text{for } \tau \in \mathbb{C}_+, z \in K_\mathbb{C}. \quad (1.5)$$

We will show that, for $f \in L^2(K, \rho_s)$ with s appropriately chosen relative to τ (cf. (1.6) below), this integral converges and defines an entire holomorphic function on $K_\mathbb{C}$ for all

$\tau \in \mathbb{C}_+$. (Note that the action of the transform does not depend on s ; at present it is used in the notation as a reminder of the domain.) Our main Theorem 1.3 then asserts that $B_{s,\tau}$ is an isometric isomorphism from $L^2(K, \rho_s)$ onto an appropriate holomorphic L^2 -space. To fix the parameters involved, we introduce the following notation.

Notation 1.2. Let $\tau = t + iu \in \mathbb{C}_+$, and let $s > 0$. The relevant parameter range for us is given by

$$s > \frac{|\tau|^2}{2\operatorname{Re} \tau}. \quad (1.6)$$

That can alternatively be written as $2s > t + u^2/t$. It is also conveniently recorded by the following constant, which will play a role in several propositions below:

$$\alpha := \frac{1}{4}(2s\operatorname{Re} \tau - |\tau|^2) = \frac{1}{4}(2st - t^2 - u^2). \quad (1.7)$$

Then Condition (1.6) is equivalent to the assumption $\alpha > 0$.

We now come to the main theorem of this paper.

Theorem 1.3. Let K be a compact type Lie group. Let $\tau \in \mathbb{C}_+$ and $s > 0$ satisfy (1.6). Let $\mu_{s,\tau}$ be the heat kernel density on $K_{\mathbb{C}}$ given in Definition 3.6 below. For $f \in L^2(K, \rho_s)$, the integral in (1.5) converges for all $z \in K_{\mathbb{C}}$, and defines a holomorphic function. The complex time Segal–Bargmann transform $B_{s,\tau}$ of (1.5) is an isometric isomorphism

$$B_{s,\tau}: L^2(K, \rho_s) \rightarrow \mathcal{H}L^2(K_{\mathbb{C}}, \mu_{s,\tau}).$$

In the case that the group K is compact, the heat kernel ρ_s on K converges weakly to the Haar measure as $s \rightarrow \infty$. A version of the Segal–Bargmann isomorphism holds in this context as well.

Theorem 1.4. Let K be compact. For any $\tau \in \mathbb{C}_+$, $B_{\infty,\tau}$ (defined by the same integral formula (1.5)) is an isometric isomorphism

$$B_{\infty,\tau}: L^2(K) \rightarrow \mathcal{H}L^2(K_{\mathbb{C}}, \nu_t)$$

where $t = \operatorname{Re} \tau$ and ν_t is the K -averaged heat kernel density on $K_{\mathbb{C}}$ given in Definition 3.7 below.

Both Theorems 1.3 and 1.4 are proved in Section 4.

1.4. A Sketch of the Proof. Let us give a heuristic proof of the isometricity portion of Theorem 1.3 here, in the Euclidean case $K = \mathbb{R}^k$, for motivation. By (1.5), if we restrict to real time $\tau = t > 0$ and look at the transform $(B_{s,t}f)(x)$ at a point $x \in \mathbb{R}^k$, we simply have $(B_{s,t}f)(x) = \int_{\mathbb{R}^k} f(y)\rho_t(x-y)dy$; in other words, restricted to real time and K , $B_{s,t}f$ is just the heat operator applied to f , $B_{s,t}f = e^{\frac{t}{2}\Delta}f$ where Δ is the standard Laplacian on \mathbb{R}^k (cf. (2.12)). Therefore, in general the transform can be described as “apply the heat operator, then analytically continue in space and time”. But if the test function f itself already possesses a holomorphic extension $f_{\mathbb{C}}$ to all of \mathbb{C}^k (e.g. if f is a polynomial), then at least informally we should have

$$B_{s,\tau}f = e^{\frac{\tau}{2}\Delta}f_{\mathbb{C}},$$

where now Δ (the sum squares of the \mathbb{R}^k -derivatives) is acting on functions on \mathbb{C}^k .

Let $F = B_{s,\tau}f$; we need to compute $|F|^2 = F\bar{F}$. Since $f_{\mathbb{C}}$ is holomorphic, we have $\frac{\partial}{\partial x_j}f_{\mathbb{C}} = \frac{\partial}{\partial z_j}f_{\mathbb{C}}$, and so $\Delta f_{\mathbb{C}} = \sum_{j=1}^k \frac{\partial^2}{\partial z_j^2}f_{\mathbb{C}} := \partial^2 f_{\mathbb{C}}$; similarly $\Delta \bar{f}_{\mathbb{C}} = \sum_{j=1}^k \frac{\partial^2}{\partial \bar{z}_j^2}\bar{f}_{\mathbb{C}} :=$

$\bar{\partial}^2 \bar{f}_{\mathbb{C}}$. Again, since $f_{\mathbb{C}}$ is holomorphic and $\bar{f}_{\mathbb{C}}$ is antiholomorphic, $\partial^2 \bar{f}_{\mathbb{C}} = 0 = \bar{\partial}^2 f_{\mathbb{C}}$; so we have

$$(F\bar{F}) = (e^{\frac{\tau}{2}\partial^2} f_{\mathbb{C}})(e^{\frac{\bar{\tau}}{2}\bar{\partial}^2} \bar{f}_{\mathbb{C}}) = e^{(\frac{\tau}{2}\partial^2 + \frac{\bar{\tau}}{2}\bar{\partial}^2)} f_{\mathbb{C}} \bar{f}_{\mathbb{C}}. \quad (1.8)$$

Now, we measure f in $L^2(\mathbb{R}^k, \rho_s)$; setting $x = 0$ in the convolution formula (2.12) defining the heat operator, and using the symmetry $\rho_s(-y) = \rho_s(y)$, we can compute

$$\|f\|_{L^2(\mathbb{R}^k, \rho_s)}^2 = \int_{\mathbb{R}^k} |f(y)|^2 \rho_s(0-y) dy = (e^{\frac{s}{2}\Delta} |f|^2)(0) = (e^{\frac{s}{2}\Delta} |f_{\mathbb{C}}|^2)(0). \quad (1.9)$$

Similarly, we measure F in $L^2(\mathbb{C}^k, \mu_{s,\tau})$, where $\mu_{s,\tau}$ is the heat kernel for some yet-to-be-determined elliptic operator $\Delta_{s,\tau}$ on \mathbb{C}^k , meaning

$$\|F\|_{L^2(\mathbb{C}^k, \mu_{s,\tau})}^2 = (e^{\frac{1}{2}\Delta_{s,\tau}} |F|^2)(0). \quad (1.10)$$

Combining (1.8) and (1.10), and commuting partial derivatives to combine the exponentials, we therefore have

$$\|B_{s,\tau} f\|_{L^2(\mathbb{C}^k, \mu_{s,\tau})}^2 = (e^{\frac{1}{2}\Delta_{s,\tau} + \frac{\tau}{2}\partial^2 + \frac{\bar{\tau}}{2}\bar{\partial}^2} |f_{\mathbb{C}}|^2)(0). \quad (1.11)$$

Comparing (1.9) with (1.11), we see that to prove the isometricity in Theorem 1.3, it suffices to have

$$s\Delta = \Delta_{s,\tau} + \tau\partial^2 + \bar{\tau}\bar{\partial}^2.$$

Expressing the operators ∂^2 and $\bar{\partial}^2$ in terms of real partial derivatives, we can then solve for $\Delta_{s,\tau}$ and find that

$$\Delta_{s,\tau} = \sum_{j=1}^k \left[\left(s - \frac{t}{2} \right) \frac{\partial^2}{\partial x_j^2} + \frac{t}{2} \frac{\partial^2}{\partial y_j^2} - u \frac{\partial^2}{\partial x_j \partial y_j} \right].$$

It is easy to check that this operator is elliptic precisely in the given parameter range (1.6).

For a general Lie group K of compact type, we replace the partial derivatives in the preceding argument with left-invariant vector fields. The heuristic argument then goes through unchanged, *except* that we must remember that left-invariant vector fields do not, in general, commute. Thus, we must also verify that the particular operators involved in the calculation do, in fact, commute, allowing us to combine the exponents as above. For this, we need to use an inner product on the Lie algebra of K that is Ad-invariant; this is the reason for the assumption that K be of compact type.

Most of this paper is devoted to making the above argument rigorous. The key is to introduce a dense subspace (consisting of matrix entries, cf. Section 2.7) of the domain Hilbert space on which integration against the heat kernel can rigorously be computed by a power series in the relevant Laplacian. This argument can be found in Section 4, with the necessary background about heat kernel analysis on Lie groups in Section 2, and the analysis of the heat kernel $\mu_{s,\tau}$ and its generator $\Delta_{s,\tau}$ in Section 3.

The operator $\Delta_{s,\tau}$ was the starting point for the current investigation. It is the Laplacian for a left-invariant Riemannian metric on $K_{\mathbb{C}}$ for which the corresponding inner product on the Lie algebra is invariant under the Adjoint action of K . While the Lie algebra of the complexified Lie group $K_{\mathbb{C}}$ does not possess an Ad-invariant inner product, it does possess many Ad(K)-invariant inner products. These are the most natural from the perspective of diffusion processes, particularly in high dimension (cf. [21]). In fact, there is a natural three (real) parameter family of Ad(K)-invariant inner products on $\text{Lie}(K_{\mathbb{C}})$ (see (3.10) for the relation to the Segal–Bargman transform parameters s and $\tau = t + iu$). In the case that K is simple, this is a complete characterization of all such invariant inner products; this is the statement of Theorem 3.1 below. It was this fact that led the authors backward to discover

the complex time Segal–Bargmann transform, which is therefore a natural completion of the versions of the transform previously introduced by Segal, Bargmann, and the first two authors of the present paper.

1.5. Motivation. The Segal–Bargmann transform $B_{s,\tau}f$ is computed by integration of f against the functions

$$\chi_\tau^z(x) := \rho_{\mathbb{C}}(\tau, x^{-1}z). \quad (1.12)$$

These functions may be thought of as “coherent states” on K . In the \mathbb{R}^1 case, coherent states are often defined as minimum uncertainty states, namely those giving equality in the classic Heisenberg uncertainty principle. There is, however, a stronger form of the uncertainty principle, due to Schrödinger [26], which says that

$$(\Delta_\chi X)^2 (\Delta_\chi P)^2 \geq \frac{\hbar^2}{4} + |\text{Cov}_\chi(X, P)|^2, \quad (1.13)$$

where $\text{Cov}_\chi(X, P) := \langle (XP + PX)/2 \rangle_\chi - \langle X \rangle_\chi \langle P \rangle_\chi$ is the quantum covariance.

States that give equality in (1.13) are Gaussian wave packets, but where the quadratic term in the exponent can be complex, as follows:

$$\chi(x) = C \exp\{iax^2 - b(x - c)^2 + idx\} \quad (1.14)$$

with $a, b, c, d \in \mathbb{R}$ and $b > 0$. This class of states is actually more natural than the usual ones with $a = 0$, because the collection of states of the form (1.14) is invariant under the metaplectic representation, that is, the natural (projective) unitary action of the group of symplectic linear transformations of \mathbb{R}^2 .

If we specialize the states in (1.12) to the \mathbb{R}^n case, we find that they are Gaussian wave packets, and that if $\text{Im } \tau \neq 0$ then the quadratic part of the exponent is complex. We see, then, that allowing the time-parameter in the Segal–Bargmann transform to be complex amounts to considering a larger and more natural family of coherent states.

In the $s \rightarrow \infty$ version of the transform $B_{\infty,\tau}$ of Theorem 1.4 (which, in the real time case $\tau = t > 0$, was referred to as the C -version of the Segal–Bargmann transform C_t in [17]), the domain Hilbert space is $L^2(K)$ (with respect to Haar measure on K). In that case, it is possible to replace the time parameter $t > 0$ with $t + iu$ with $t > 0$ and u an arbitrary real number. The modified map C_{t+iu} is still a unitary map between the same two Hilbert spaces, because $e^{iu\Delta/2}$ is a unitary map of $L^2(K, dx)$ to itself. This possibility has been exploited, for example, in the papers [11] and [12] of C. Florentino, J. Mourão, and J. Nunes on the quantization of nonabelian theta functions on $\text{SL}(n, \mathbb{C}) = \text{SU}(n)_{\mathbb{C}}$. The authors show that these functions arise as the image of certain distributions on $\text{SU}(n)$ under the heat operator, evaluated at a complex time, and use the Segal–Bargmann transform in the complexification process.

Meanwhile, the Segal–Bargmann transform for K is related to the study of complex structures on the cotangent bundle $T^*(K)$. There is a natural one-parameter family of “adapted complex structures” on $T^*(K)$ arising from a general construction of Guillemin–Stenzel [13, 14] and Lempert–Szöke [23, 30]. Motivated by ideas of Thiemann [31], the second author and W. Kirwin in [19] showed that these structures arise from the “imaginary-time geodesic flow” on $T^*(K)$. The Segal–Bargmann transform can then be understood [9, 10, 18] as a quantum counterpart of the construction in [19].

As observed in [24], the adapted complex structures on $T^*(K)$ extend to a two-parameter family, by including both a real and an imaginary part to the time-parameter in the geodesic flow in [19]. The corresponding quantum construction has been done in [25] and can be

thought of as adding a complex parameter to the C -version of the Segal–Bargmann transform for K . (Compare work of Kirwin and Wu [22] in the \mathbb{R}^k case.) The present paper then extends the complex-time transform to its most natural range, in which the domain Hilbert space is taken to be L^2 of K with respect to a heat kernel measure.

2. BACKGROUND AND NOTATION

Let G be a real Lie group, e denote the identity element of G , $\iota : G \rightarrow G$ be the inversion map, $\iota(x) = x^{-1}$ for all $x \in G$, and for any $g \in G$ let $L_g, R_g : G \rightarrow G$ be the left and right translation by g maps defined by $L_g(x) = gx$ and $R_g(x) = xg$ for all $x \in G$. We now choose once and for all a right Haar measure, $\lambda = \lambda_G$, on G and usually simply write dx for $d\lambda(x)$ and $L^2(G)$ for $L^2(G, \lambda)$. The Lie algebra of G is taken to be $\mathfrak{g} := T_e G$ and to each $V \in \mathfrak{g}$ we let \tilde{V} be the unique left invariant vector field on G such that $\tilde{V}(e) = V$, i.e. $\tilde{V}(g) = L_{g*} V$ for all $g \in G$. As usual, for $g \in G$, Ad_g denotes the endomorphism of \mathfrak{g} given by $\text{Ad}_g = (C_g)_*$, where $C_g = L_g R_{g^{-1}}$ is the conjugation map on G . Then $\text{Ad} : G \rightarrow \text{GL}(\mathfrak{g})$, $g \mapsto \text{Ad}_g$ is a Lie group homomorphism. Its derivative is $\text{ad} = \text{Ad}_*$, which is a Lie algebra homomorphism from \mathfrak{g} to $\text{End}(\mathfrak{g})$. It is given explicitly by $\text{ad}_X(Y) = [X, Y]$.

2.1. Complex Lie Groups and Complexification. Let M be a complex manifold, of complex dimension d . For $U \subseteq M$ open, let $\mathcal{H}(U)$ denote the space of holomorphic functions $U \rightarrow \mathbb{C}$. The complex structure provides a smooth section J of the endomorphism bundle $\text{End}(TM)$ defined thus: in any holomorphic chart $\zeta : U \subseteq M \rightarrow \mathbb{C}^d$ and any $v \in TU$,

$$(Jv)(\zeta) = i(v(\zeta)).$$

(The existence of a holomorphic atlas makes J globally well-defined.) It follows that, in fact,

$$(Jv)(f) = i(v(f)) \text{ for all } v \in TU \text{ and } f \in \mathcal{H}(U), \quad (2.1)$$

and this characterizes J uniquely. It follows immediately that $J^2 = -\text{Id}$, so it is an almost complex structure. It is also integrable (in the sense of Fröbenius), which in this context can be characterized by the following commutator relation (the vanishing of the Nijenhuis tensor):

$$[JV, JW] = [V, W] + J[JV, W] + J[V, JW] \quad (2.2)$$

for all smooth local vector fields V, W on M .

Lemma 2.1. *If G is a complex Lie group and $V \in \mathfrak{g}$, then $\widetilde{JV}f = i(\tilde{V}f)$ for all (locally) holomorphic functions f on G .*

Proof. Let $U \subseteq G$ be open, and let $f \in \mathcal{H}(U)$. For any point $g \in U$,

$$(\tilde{V}f)(g) = V(f \circ L_g).$$

Since $f \circ L_g$ is holomorphic on $L_{g^{-1}}(U)$, applying (2.1) we have

$$(\widetilde{JV}f)(g) = JV(f \circ L_g) = iV(f \circ L_g) = i(\tilde{V}f)(g).$$

□

Remark 2.2. It is worth noting that the output function $\tilde{V}f$ is still locally holomorphic, for any left-invariant vector field \tilde{V} . Indeed,

$$(\tilde{V}f)(g) = \left. \frac{d}{dt} \right|_{t=0} f(g \exp tV)$$

and $g \mapsto f(g \exp tV)$ is holomorphic for each t .

Corollary 2.3. *For $V, W \in \mathfrak{g}$, $[JV, W] = J[V, W]$.*

Proof. Let $U \subseteq G$ be a neighborhood of the identity $e \in G$, and let $f \in \mathcal{H}(U)$. Then

$$\begin{aligned} [JV, W]f &= (\widetilde{JVW} - \widetilde{VJV})f|_e \\ &= i\widetilde{V}(\widetilde{W}f)|_e - \widetilde{W}(i\widetilde{V}f)|_e \\ &= i(\widetilde{V}\widetilde{W} - \widetilde{W}\widetilde{V})f|_e \\ &= i[V, W]f = (J[V, W])f. \end{aligned}$$

As this holds for all $f \in \mathcal{H}(U)$, by taking f to be the coordinate functions of a holomorphic chart at e , we conclude that $[JV, W] = J[V, W]$, as claimed. \square

Remark 2.4. Corollary 2.3, together with the relation $J^2 = -\text{Id}$, quickly verifies the Nijenhuis identity (2.2). Turning this around: if G is any real Lie group whose Lie algebra has a complex structure, this defines a section J of $\text{End}(TG)$ (by left translating any vector to the identity) which satisfies $J^2 = -\text{Id}$ and therefore satisfies the Nijenhuis condition. This gives a proof that any Lie group whose Lie algebra has a complex structure is, in fact, a complex Lie group. (It is also easy to see this directly. A complex vector space structure on \mathfrak{g} induces a complex manifold structure on G , for example by constructing a holomorphic atlas using exponential coordinates.)

We also note that the complex structure J commutes with the adjoint maps.

Lemma 2.5. *With G , \mathfrak{g} , and J as above,*

$$\text{Ad}_g J = J \text{Ad}_g \quad \text{and} \quad \text{ad}_V J = J \text{ad}_V$$

for all $g \in G$ and $V \in \mathfrak{g}$.

Proof. For each $g \in G$, the conjugation map $C_g = L_g R_{g^{-1}}: G \rightarrow G$ is holomorphic and fixes e ; thus its derivative Ad_g is complex linear; in particular, this means that Ad_g commutes with J , as desired. Applying this with $g = e^{tV}$ and differentiating at $t = 0$ shows that ad_V commutes with J as well. \square

Let K be a connected real Lie group. Say that a complex Lie group G is a **complexification** of K if the following universal property holds: there is a real Lie group homomorphism $\phi: K \rightarrow G$ such that, if H is any complex Lie group and $\Phi: K \rightarrow H$ is a real Lie group homomorphism, there exists a unique holomorphic homomorphism $\Phi_{\mathbb{C}}: G \rightarrow H$ through which ϕ factors as $\Phi = \Phi_{\mathbb{C}} \circ \phi$:

$$\begin{array}{ccc} K & \xrightarrow{\phi} & G \\ & \searrow \Phi & \swarrow \Phi_{\mathbb{C}} \\ & H & \end{array}$$

Proposition 2.6 ([16], Lemma 3). *If K is a compact, connected Lie group with Lie algebra \mathfrak{k} , there exists a complexification of K , and it is unique up to isomorphism; we refer to it as $K_{\mathbb{C}}$. Moreover: $K_{\mathbb{C}}$ is connected, the canonical homomorphism $\phi: K \rightarrow K_{\mathbb{C}}$ is injective, and its image is a maximal compact subgroup of $K_{\mathbb{C}}$. Also: the Lie algebra $\mathfrak{k}_{\mathbb{C}}$ of $K_{\mathbb{C}}$ is the complexification of the Lie algebra of K : $\mathfrak{k}_{\mathbb{C}} = \mathfrak{k} \otimes_{\mathbb{R}} \mathbb{C} = \mathfrak{k} \oplus J\mathfrak{k}$.*

Remark 2.7. It is worth noting that complexifications can exist beyond the compact type category, but the injectivity of the inclusion is not guaranteed, and the identification of the complexified Lie algebra can be more cumbersome. See the following examples.

Example 2.8. The compact Lie groups $\mathrm{SO}(n, \mathbb{R})$, $\mathrm{SU}(n)$, and $\mathrm{U}(n)$ have the following complexifications:

$$\mathrm{SO}(n, \mathbb{R})_{\mathbb{C}} = \mathrm{SO}(n, \mathbb{C}), \quad \mathrm{SU}(n)_{\mathbb{C}} = \mathrm{SL}(n, \mathbb{C}), \quad \mathrm{U}(n)_{\mathbb{C}} = \mathrm{GL}(n, \mathbb{C}).$$

The real Lie groups $\mathrm{GL}(n, \mathbb{R})$ and $\mathrm{GL}(n, \mathbb{C})$ both have complexifications:

$$\mathrm{GL}(n, \mathbb{R})_{\mathbb{C}} = \mathrm{GL}(n, \mathbb{C}), \quad \mathrm{GL}(n, \mathbb{C})_{\mathbb{C}} = \mathrm{GL}(n, \mathbb{C}) \times \mathrm{GL}(n, \mathbb{C}).$$

The preceding example demonstrates a subtle point: complexifying a Lie group always treats it as a real Lie group, even if it happens to already be complex. In particular, as the case $\mathrm{GL}(n, \mathbb{C})_{\mathbb{C}}$ demonstrates, the identification $\mathfrak{g}_{\mathbb{C}} = \mathfrak{g} \oplus J\mathfrak{g}$ does not hold generally. Instead, one always has $\mathfrak{g}_{\mathbb{C}} \cong \mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C}$.

2.2. Compact Type Lie groups.

Definition 2.9. Let G be a Lie group, and let $K \subseteq G$ be a Lie subgroup. An inner product $\langle \cdot, \cdot \rangle_{\mathfrak{g}}$ on \mathfrak{g} is $\mathrm{Ad}(K)$ -invariant if, for all $V_1, V_2 \in \mathfrak{g}$ and all $k \in K$,

$$\langle \mathrm{Ad}_k V_1, \mathrm{Ad}_k V_2 \rangle_{\mathfrak{g}} = \langle V_1, V_2 \rangle_{\mathfrak{g}}.$$

If the inner product is $\mathrm{Ad}(G)$ -invariant, we simply call it Ad -invariant. A group whose Lie algebra possesses an Ad -invariant inner product is called **compact type**.

Every compact Lie group possesses Ad -invariant inner products — simply average any inner product over the Haar measure on the group — thus explaining the terminology “compact type”. The simplest examples are Lie subgroups of $\mathrm{U}(n)$; for any such group K , the Hilbert-Schmidt inner product $\langle V_1, V_2 \rangle = \mathrm{Tr}(V_1 V_2^*)$ is Ad -invariant. (In the case of the simple group $\mathrm{SU}(n)$, this is, up to scale, the only Ad -invariant inner product.)

Note that the existence of an Ad -invariant inner product means that there is a basis in which Ad_g is unitary for all g ; in particular, this means $\det(\mathrm{Ad}_g) = 1$ for all g in this case. It follows (taking $g = e^{tX}$ and taking $\frac{d}{dt}|_{t=0}$) that ad_X is skew-Hermitian, and in particular $\mathrm{Tr}(\mathrm{ad}_X) = 0$ in this case.

It turns out that the presence of an Ad -invariant inner product nearly forces the group to be compact.

Proposition 2.10 ([18], Theorem 2.2). *If H is a compact type Lie group with a specified Ad -invariant inner product, then H is isometrically isomorphic to a Cartesian product $H \cong K \times \mathbb{R}^k$ for some compact Lie group K .*

Remark 2.11. It follows readily from Proposition 2.10 and Proposition 2.6 that connected compact type Lie groups also have complexifications: if $H \cong K \times \mathbb{R}^k$, then $H_{\mathbb{C}} = K_{\mathbb{C}} \times \mathbb{C}^k$. The inclusion map $H \rightarrow H_{\mathbb{C}}$ is still injective, and we still have $\mathfrak{h}_{\mathbb{C}} = \mathfrak{h} \oplus J\mathfrak{h}$ in this case (where \mathfrak{h} is the Lie algebra of H).

2.3. The Modular Function. Recall that the modular function, $m : G \rightarrow (0, \infty)$, is the continuous (in fact smooth) group homomorphism determined by $(L_g)_* \lambda = m(g)\lambda$ for all $g \in G$. It is easy to verify that both $\iota_* \lambda$ and $m\lambda$ are left invariant Haar measures on G and hence $\iota_* \lambda = Cm\lambda$ for some $C > 0$. Applying ι_* to the equation $\iota_* \lambda = Cm\lambda$ using $\iota^{-1} = \iota$, $\iota_*(m\lambda) = m \circ \iota^{-1} \cdot \iota_* \lambda$, and $m \cdot m \circ \iota = 1$ by the homomorphism property of ι

one easily deduces that $\lambda = \iota_*^2 \lambda = C^2 \lambda$ from which it follows that $C = 1$, i.e. $\iota_* \lambda = m \lambda$. The above remarks may be summarized by the identities;

$$\begin{aligned} \int_G f(gx) dx &= m(g) \int_G f(x) dx \quad \forall g \in G \text{ and} \\ \int_G f(x^{-1}) dx &= \int_G f(x) m(x) dx \end{aligned}$$

which hold for all nonnegative $f \in C_c(G)$.

A Lie group whose modular function is constantly equal to 1 is called *unimodular*. For example, compact Lie groups are always unimodular; in fact, $m|_K \equiv 1$ for any compact subgroup, $K \subset G$. This follows from the fact that $m(K)$ has to be a compact subgroup of $(0, \infty)$ and there is only one such subgroup, namely $\{1\}$. We will now explore a different explanation for this fact – one that readily generalizes to compact type Lie groups (and their complexifications).

Lemma 2.12. *The modular function is given explicitly as*

$$m(g) = \det(\text{Ad}_{g^{-1}}) \quad \forall g \in G. \quad (2.3)$$

Proof. Let v denote the volume form (i.e. top-rank differential form) associated to λ . Then the volume form associated to $L_{g*} \lambda$ is $L_{g^{-1}}^* v$, and hence $L_{g^{-1}}^* v = m(g)v$. Now fixing any basis $\{X_i\}$ of \mathfrak{g} , we have

$$\begin{aligned} (L_{g^{-1}}^* v)(X_1, \dots, X_d) &= v(L_{g^{-1}*} X_1, \dots, L_{g^{-1}*} X_d) \\ &= v_e(R_{g*} L_{g^{-1}*} X_1, \dots, R_{g*} L_{g^{-1}*} X_d) \\ &= v_e(\text{Ad}_{g^{-1}} X_1, \dots, \text{Ad}_{g^{-1}} X_d) \\ &= \det(\text{Ad}_{g^{-1}}) v_e(X_1, \dots, X_d) \end{aligned}$$

which yields (2.3). \square

Corollary 2.13. *If G is a Lie group, then*

- (1) G unimodular iff $\det(\text{Ad}_g) = 1$ for all $g \in G$.
- (2) If G is unimodular, then $\text{Tr}(\text{ad}_A) = 0$ for all $A \in \mathfrak{g}$.
- (3) If $\text{Tr}(\text{ad}_A) = 0$ for all $A \in \mathfrak{g}$ and G is connected, then G is unimodular.
- (4) We always have

$$m(e^{-A}) = \det(\text{Ad}_{e^A}) = e^{\text{Tr}(\text{ad}_A)}. \quad (2.4)$$

Proof. Item (1) is immediate from Lemma 2.12. For item (2), if G is unimodular then $\text{Ad}_{e^t A}$ has constant determinant, and so

$$0 = \frac{d}{dt} \Big|_{t=0} \det(\text{Ad}_{e^{tA}}) = \frac{d}{dt} \Big|_{t=0} \det(e^{t \text{ad}_A}) = \text{Tr}(\text{ad}_A)$$

for all $A \in \mathfrak{g}$. For the converse, if $g(t)$ is a curve in G such that $g(0) = e$, then

$$\begin{aligned} \frac{d}{dt} \det(\text{Ad}_{g(t)}) &= \frac{d}{ds} \Big|_{s=0} \det(\text{Ad}_{g(t)g(t)^{-1}g(t+s)}) \\ &= \det(\text{Ad}_{g(t)}) \frac{d}{ds} \Big|_{s=0} \det(\text{Ad}_{g(t)^{-1}g(t+s)}) \\ &= \det(\text{Ad}_{g(t)}) \text{Tr}(\text{ad}_{L_{g(t)^{-1}*} \dot{g}(t)}) \end{aligned}$$

and hence

$$\det(\text{Ad}_{g(1)}) = \exp \left(\int_0^1 \text{Tr}(\text{ad}_{L_{g(t)}^{-1} \dot{g}(t)}) dt \right) \quad (2.5)$$

wherein we have used $\det(\text{Ad}_{g(0)}) = \det(\text{Id}) = 1$. In particular, if $\text{Tr}(\text{ad}_A) = 0$ for all $A \in \mathfrak{g}$ then $m(g) = 1$ on the connected component of the identity in G , verifying item (3). Taking $g(t) = e^{tA}$ in (2.5) gives (2.4), verifying (4) and completing the proof. \square

It now follows that all the Lie groups we deal with in this paper (compact type groups and their complexifications) are unimodular.

Corollary 2.14. *If K is a compact type Lie group, then K is unimodular. If, in addition, K is connected, then $K_{\mathbb{C}}$ is unimodular.*

Proof. As noted above, if K is compact type, then Ad_k has determinant 1 for all $k \in K$, and so K is unimodular by Corollary 2.13(1). Now, for any $A \in \mathfrak{k}_{\mathbb{C}}$, there are $X, Y \in \mathfrak{k}$ with $A = X + iY$. Then $\text{ad}_A = [X + iY, \cdot] = [X, \cdot] + i[Y, \cdot] = \text{ad}_X + i\text{ad}_Y$. By Corollary 2.13(2), ad_X and ad_Y both have trace 0, and hence so does ad_A . Since $K_{\mathbb{C}}$ is connected, it follows from Corollary 2.13(3) that $K_{\mathbb{C}}$ is unimodular. \square

2.4. Left Invariant Integral Operators. Let $BC(G)$ denote the bounded continuous complex valued functions on G , and for $g \in G$ let \hat{L}_g and \hat{R}_g be the operators on $BC(G)$ defined by $\hat{L}_g f = f \circ L_g$ and $\hat{R}_g f = f \circ R_g$ respectively. We also consider \hat{R}_g as a unitary operator on $L^2(G)$.

Notation 2.15. *If γ is a complex measure on G we let $\Gamma_{\gamma} = \int_G \hat{R}_y \gamma(dy)$ which may be viewed as a bounded linear operator on either $BC(G)$ or $L^2(G)$ with operator norm bounded by $|\gamma|(G)$ in either case. (Here $|\gamma|$ denote the total variation measure of γ .)*

The next lemma, whose simple proof is left to the reader, explains our interest in operators of the form Γ_{γ} .

Lemma 2.16. *Suppose that $T: BC(G) \rightarrow BC(G)$ is a linear operator of the form*

$$(Tf)(x) = \int_G f(y) \gamma(x, dy)$$

where $\gamma(x, \cdot)$ is a complex measure on G for each $x \in G$. Then the following are equivalent:

- (1) T is left invariant; i.e. $[T, \hat{L}_g] = 0$ for all $g \in G$.
- (2) $\gamma(x, \cdot) = (L_x)_* \gamma(e, \cdot)$ for all $x \in G$.
- (3) $T = \Gamma_{\gamma(e, \cdot)} = \int_G \hat{R}_y \gamma(e, dy)$.

Proposition 2.17. *Let γ be a complex measure on G , and let $g \in G$. Then the following are equivalent.*

- (1) $[\Gamma_{\gamma}, \hat{R}_g] = 0$.
- (2) $R_{g*} \gamma = L_{g*} \gamma$.
- (3) $(\text{Ad}_{g^{-1}})_* \gamma = \gamma$.

Moreover, if γ is absolutely continuous relative to Haar measure, i.e. $\gamma = \rho \lambda$ for some function ρ , then any of the above conditions are equivalent to ρ satisfying

$$\rho(gxg^{-1}) = m(g)\rho(x) \text{ for } \lambda\text{-a.e. } x.$$

Proof. We first note that $\hat{R}_g \hat{R}_x = \hat{R}_{gx}$ and therefore,

$$\Gamma_\gamma \hat{R}_g = \int_G \hat{R}_y \hat{R}_g \gamma(dy) = \int_G \hat{R}_{yg} \gamma(dy) = \Gamma_{R_{g*} \gamma}$$

and similarly $\hat{R}_g \Gamma_\gamma = \Gamma_{L_{g*} \gamma}$. Therefore $[\Gamma_\gamma, \hat{R}_g] = 0$ iff $R_{g*} \gamma = L_{g*} \gamma$ which happens iff $(\text{Ad}_{g^{-1}})_* \gamma = L_{g^{-1}*} R_{g*} \gamma = \gamma$.

If $\gamma = \rho \lambda$ then

$$\begin{aligned} R_{g*} \gamma &= \rho \circ R_{g^{-1}} \cdot R_{g*} \lambda = \rho \circ R_{g^{-1}} \lambda, \quad \text{and} \\ L_{g*} \gamma &= \rho \circ L_{g^{-1}} \cdot L_{g*} \lambda = \rho \circ L_{g^{-1}} \cdot m(g) \lambda. \end{aligned}$$

Therefore $R_{g*} \gamma = L_{g*} \gamma$ iff $\rho \circ R_{g^{-1}} = m(g) \rho \circ L_{g^{-1}} \lambda - \text{a.e.}$, i.e. iff

$$\rho(xg^{-1}) = m(g) \rho(g^{-1}x) \text{ for } \lambda - \text{a.e. } x.$$

The result now follows by making the substitution $x \rightarrow gx$ in the above equation. \square

The straightforward proof of the next proposition is also left to the reader.

Proposition 2.18. *Let γ be a complex measure on G and now view Γ_γ as a bounded operator on $L^2(G)$. Then $\Gamma_\gamma^* = \Gamma_{\iota_* \bar{\gamma}}$. It follows that Γ_γ is self-adjoint iff $\iota_* \bar{\gamma} = \gamma$, i.e. iff*

$$\int_G f(x^{-1}) \bar{\gamma}(dx) = \int_G f(x) \gamma(dx) \quad \forall f \in BC(G).$$

In particular, if γ is a positive measure then $\Gamma_\gamma^ = \Gamma_\gamma$ iff $\iota_* \gamma = \gamma$.*

We now come to group convolution, which plays a central role in heat kernel analysis on Lie groups.

Definition 2.19. *Let γ_1 and γ_2 be two complex measures on G . We define $\gamma_1 * \gamma_2 := M_*(\gamma_1 \otimes \gamma_2)$ where $M: G \times G \rightarrow G$ is the multiplication map $M(x, y) = xy$ for all $x, y \in G$. Alternatively stated, $\gamma_1 * \gamma_2$ is the unique complex measure on G such that*

$$\int_G f d(\gamma_1 * \gamma_2) = \int_{G \times G} f(xy) \gamma_1(dx) \gamma_2(dy) \quad \forall f \in BC(G).$$

Example 2.20. If $\gamma_j = \rho_j \lambda$ for some $\rho_j \in L^2(G, \lambda)$, then it is easy to verify that $\gamma_1 * \gamma_2$ is absolutely continuous relative to λ and

$$\frac{d(\gamma_1 * \gamma_2)}{d\lambda}(x) = \int_G \rho_1(y) \rho_2(y^{-1}x) dy.$$

Lemma 2.21. *If γ_1 and γ_2 are two complex measures on G , then $\Gamma_{\gamma_1} \Gamma_{\gamma_2} = \Gamma_{\gamma_1 * \gamma_2}$.*

Proof.

$$\begin{aligned} \Gamma_{\gamma_1} \Gamma_{\gamma_2} &= \int_G \hat{R}_x \gamma_1(dx) \int_G \hat{R}_y \gamma_2(dy) \\ &= \int_G \hat{R}_{xy} \gamma_1(dx) \gamma_2(dy) = \int_G \hat{R}_z (\gamma_1 * \gamma_2)(dz). \end{aligned}$$

\square

2.5. Laplacians. We now introduce left invariant Laplacian operators, which in this context will mean any sum of squares of left invariant vector fields.

Notation 2.22. To each subspace $V \subset \mathfrak{g}$ equipped with an inner product $\langle \cdot, \cdot \rangle_V$, let

$$\Delta_V = \sum_{a=1}^{d_V} \tilde{Z}_a^2$$

where $d_V = \dim_{\mathbb{R}} V$ and $\{Z_a\}_{a=1}^{d_V}$ is an orthonormal basis for $(V, \langle \cdot, \cdot \rangle_V)$.

By construction Δ_V is a left invariant differential operator on G which (as the next lemma shows) is well defined independent of basis.

Lemma 2.23 (Left Invariant Laplacians). *Continuing the notation above, let $\{X_j\}_{j=1}^{d_V}$ be any basis for V , and define $q_{ij} := \langle X_i, X_j \rangle_V$ (the Gram matrix). If q^{-1} is the matrix inverse to q , then (as differential operators on $C^2(G)$),*

$$\sum_{i,j} q_{ij}^{-1} \tilde{X}_i \tilde{X}_j = \sum_a \tilde{Z}_a^2$$

where i, j, a all run over $\{1, 2, \dots, d_V\}$. As a corollary we see that these expressions are basis independent, i.e. the operators above are associated purely to the metric $\langle \cdot, \cdot \rangle_V$.

Proof. If we let $A_{\ell j} := \langle Z_\ell, X_j \rangle_V$, then

$$\begin{aligned} q_{ij} &= \langle X_i, X_j \rangle_V = \sum_{\ell} \langle X_i, Z_\ell \rangle_V \langle Z_\ell, X_j \rangle_V \\ &= \sum_{\ell} A_{\ell j} A_{\ell i} = (A^\top A)_{ij} \end{aligned}$$

from which we easily conclude that $Aq^{-1}A^\top = I$. Using this identity, we find

$$\begin{aligned} \sum_{i,j} q_{ij}^{-1} \tilde{X}_i \tilde{X}_j &= \sum_{i,j,a,b} q_{ij}^{-1} \langle X_i, Z_a \rangle_V \langle X_j, Z_b \rangle_V \tilde{Z}_a \tilde{Z}_b \\ &= \sum_{i,j,a,b} q_{ij}^{-1} A_{ai} A_{bj} \tilde{Z}_a \tilde{Z}_b = \sum_{i,j,a,b} A_{ai} q_{ij}^{-1} A_{jb}^\top \tilde{Z}_a \tilde{Z}_b \\ &= \sum_{a,b} [Aq^{-1}A^\top]_{a,b} \tilde{Z}_a \tilde{Z}_b = \sum_a \tilde{Z}_a^2. \end{aligned}$$

□

So Δ_V is completely determined by the chosen inner product on V . We now see that Ad-invariance of this inner product is reflected as *right*-invariance of Δ_V (in addition to its intrinsic left-invariance).

Lemma 2.24. *Let K be a Lie subgroup of G , let $V \subset \mathfrak{g}$ be an $\text{Ad}(K)$ -invariant subspace and let $\langle \cdot, \cdot \rangle_V$ be an $\text{Ad}(K)$ -invariant inner product on V . Then Δ_V is bi-invariant: it is invariant under \hat{L}_k and \hat{R}_k for all $k \in K$.*

Proof. As Δ_V is a linear combination of left invariant differential operators, Δ_V is in fact left invariant over G ; i.e. Δ_V is invariant under $f \mapsto \hat{L}_g f = f \circ L_g$ for all $g \in G$. For the desired right invariance, we first calculate the action of a left invariant vector field \tilde{Z} on a right-translated function.

$$\begin{aligned}
\tilde{Z}(f \circ R_g)(x) &= \frac{d}{dt} \Big|_{t=0} (f \circ R_g)(xe^{tZ}) = \frac{d}{dt} \Big|_{t=0} f(xe^{tZ}g) \\
&= \frac{d}{dt} \Big|_{t=0} f(xgg^{-1}e^{tZ}g) = \frac{d}{dt} \Big|_{t=0} f(xge^{t\text{Ad}_g^{-1}Z}) = (\tilde{Z}^g f)(xg)
\end{aligned}$$

where $Z^g = \text{Ad}_g^{-1}Z$. In other words, we have shown that

$$\tilde{Z}(f \circ R_g) = (\tilde{Z}^g f) \circ R_g.$$

Thus, if $k \in K$, we have

$$\Delta_V(f \circ R_k) = \sum_{a=1}^{d_V} \tilde{Z}_a^2(f \circ R_k) = \sum_{a=1}^{d_V} \left((\tilde{Z}_a^k)^2 f \right) \circ R_k. \quad (2.6)$$

As $\langle \cdot, \cdot \rangle_V$ is $\text{Ad}(K)$ -invariant, we know that $\{Z_a^k\}_{a=1}^{d_V} = \{\text{Ad}_{k^{-1}}Z_a\}_{a=1}^{d_V}$ is still an orthonormal basis for $(V, \langle \cdot, \cdot \rangle_V)$. By Lemma 2.23,

$$\sum_{a=1}^{d_V} (\tilde{Z}_a^k)^2 = \Delta_V,$$

and thus (2.6) shows that $\Delta_V(f \circ R_k) = (\Delta_V f) \circ R_k$ as desired. \square

Corollary 2.25. *Let K be a Lie subgroup of G , let $V \subset \mathfrak{g}$ be an $\text{Ad}(K)$ -invariant subspace and let $\langle \cdot, \cdot \rangle_V$ be an $\text{Ad}(K)$ -invariant inner product on V . Then $[\Delta_V, \tilde{A}] = 0$ for all $A \in \mathfrak{k}$.*

Proof. Taking $k = e^{tA}$ in the identity

$$\Delta_V(f \circ R_k) = (\Delta_V f) \circ R_k \text{ for all } k \in K$$

proved in Lemma 2.24 and then differentiating the result at $t = 0$ shows the desired equality $\Delta_V \tilde{A}f = \tilde{A}\Delta_V f$. \square

2.6. Heat Operators and Heat Kernels. We now come to the central objects used in this paper: heat operators (i.e. heat semigroups) and their integral kernels. The first important fact is that left invariant Laplacians are always essentially self-adjoint. This is the content of the next theorem, which is well known; for completeness, we include a self-contained proof in Appendix A, which is adapted from notes due to L. Gross.

Theorem 2.26. *For any subspace $V \subseteq \mathfrak{g}$, the left invariant Laplacian Δ_V , with domain $\mathcal{D}(\Delta_V) = C_c^\infty(G)$, is essentially self-adjoint as an unbounded operator on $L^2(G)$. Moreover, its closure $\bar{\Delta}_V$ is ≤ 0 , and the associated heat operators $e^{\frac{t}{2}\bar{\Delta}_V}$ are left invariant for each $t > 0$.*

Remark 2.27. If K is a Lie subgroup of G , $V \subset \mathfrak{g}$ is an $\text{Ad}(K)$ -invariant subspace, and $\langle \cdot, \cdot \rangle_V$ is an $\text{Ad}(K)$ -invariant inner product on V , then it is a basic exercise in functional analysis to conclude that $[e^{\frac{t}{2}\bar{\Delta}_V}, \hat{R}_k] = 0$ for all $k \in K$.

Remark 2.28. We will be careful to always use the explicit closure $\bar{\Delta}_V$ when applying the heat operator defined through the spectral theorem for unbounded operators as above. In later sections, we will often work in a function space (sometimes nearly disjoint from L^2) on which the naïve power series definition of $e^{\frac{t}{2}\Delta_V} f$ converges for each f (cf. Section 2.7). In that case (and that case *only*), we use the notation $e^{\frac{t}{2}\Delta_V}$ without the explicit closure.

Notation 2.29. If $\langle \cdot, \cdot \rangle_{\mathfrak{g}}$ is an inner product on \mathfrak{g} , let $|x|_{\mathfrak{g}}$ denote the Riemannian distance from e to x in G relative to the unique left invariant Riemannian metric on G which agrees with $\langle \cdot, \cdot \rangle_{\mathfrak{g}}$ on $T_e G$.

The next theorem introduces the *heat kernel*: the integral kernel of $e^{\frac{t}{2}\Delta_{\mathfrak{g}}}$. For proofs of the fundamental properties listed here, we refer the reader to [6, Proposition 3.1, Lemmas 4.2-4.3], [7, Section 3], and the references therein.

Theorem 2.30. Let $\langle \cdot, \cdot \rangle_{\mathfrak{g}}$ be an inner product on \mathfrak{g} and let $\Delta_{\mathfrak{g}}$ be the associated Laplacian as in Notation 2.22, with $\mathcal{D}(\Delta_{\mathfrak{g}}) := C_c^\infty(G)$. Then $\Delta_{\mathfrak{g}}$ is an elliptic differential operator and there is a smooth function $(0, \infty) \times G \ni (t, g) \rightarrow \rho_t^{\Delta_{\mathfrak{g}}}(g) \in (0, \infty)$ so that

$$e^{\frac{t}{2}\Delta_{\mathfrak{g}}} = \Gamma_{\rho_t^{\Delta_{\mathfrak{g}}}\lambda} = \int_G \rho_t^{\Delta_{\mathfrak{g}}}(g) \hat{R}_g dg \quad \forall t > 0. \quad (2.7)$$

This function $\rho_t^{\Delta_{\mathfrak{g}}}$ is called the **heat kernel**. It satisfies the following properties.

- (1) The measures $\{\rho_t^{\Delta_{\mathfrak{g}}}(g) dg\}_{t>0}$ are invariant under $\iota: g \mapsto g^{-1}$ (cf. Proposition 2.18).
- (2) $\rho_t^{\Delta_{\mathfrak{g}}}$ is conservative:

$$\int_G \rho_t^{\Delta_{\mathfrak{g}}}(x) dx = 1. \quad (2.8)$$

- (3) $\{\rho_t^{\Delta_{\mathfrak{g}}}\}_{t>0}$ satisfies the semigroup property:

$$\rho_{s+t}^{\Delta_{\mathfrak{g}}}(x) = \int_G \rho_s^{\Delta_{\mathfrak{g}}}(xy^{-1}) \rho_t^{\Delta_{\mathfrak{g}}}(y) dy \quad \forall s, t > 0. \quad (2.9)$$

- (4) $(t, g) \rightarrow \rho_t^{\Delta_{\mathfrak{g}}}(g)$ satisfies the heat equation:

$$\partial_t \rho_t^{\Delta_{\mathfrak{g}}}(g) = \frac{1}{2} \Delta_{\mathfrak{g}} \rho_t^{\Delta_{\mathfrak{g}}}(g) \quad \text{for } t > 0 \text{ and } g \in G.$$

- (5) $\{\rho_t^{\Delta_{\mathfrak{g}}}\}_{t>0}$ is an approximate identity: for any $f \in C_c(G)$ and $x \in G$,

$$\lim_{t \downarrow 0} \int_G f(xy^{-1}) \rho_t^{\Delta_{\mathfrak{g}}}(y) dy = \lim_{t \downarrow 0} \int_G f(xy) \rho_t^{\Delta_{\mathfrak{g}}}(y) dy = f(x). \quad (2.10)$$

- (6) (Gaussian heat kernel bounds) There exists $\nu \in \mathbb{R}$ such that for $T > 0$ and $\varepsilon \in (0, 1]$, there is a constant $C(T, \varepsilon)$ such that, for $0 < s \leq T$ and $g \in G$,

$$\rho_s^{\Delta_{\mathfrak{g}}}(g) \leq C(T, \varepsilon) s^{-d} \exp\{-(|g| - \nu s)^2 / (1 + \varepsilon)s\}. \quad (2.11)$$

Moreover, if G is unimodular, these estimates hold with $\nu = 0$.

- (7) (Exponential integrability) For all $\kappa > 0$ and compact intervals $\mathcal{J} \subset (0, \infty)$,

$$\int_G e^{\kappa|g|} \max_{s \in \mathcal{J}} \rho_s^{\Delta_{\mathfrak{g}}}(g) dg < \infty.$$

- (8) (Concentration at the identity) For any $s_1 > 0$,

$$\lim_{t \downarrow 0} \int_{|g| \geq 1} e^{|g|^2/s_1} \rho_t^{\Delta_{\mathfrak{g}}}(g) dg = 0.$$

Remark 2.31. Equation (2.7) says (together with item (1)) that, for $f \in L^2(G)$,

$$\left(e^{\frac{t}{2}\Delta_{\mathfrak{g}}} f\right)(x) = \int_G \rho_t^{\Delta_{\mathfrak{g}}}(g^{-1}) f(xg) dg.$$

Making the substitution $g \mapsto x^{-1}g$ gives the alternate formula

$$\left(e^{\frac{t}{2}\bar{\Delta}_{\mathfrak{g}}}f\right)(x) = m(x) \int_G f(g) \rho_t^{\Delta_{\mathfrak{g}}}(g^{-1}x) dg.$$

In particular, if G is unimodular, we have the more familiar formula

$$\left(e^{\frac{t}{2}\bar{\Delta}_{\mathfrak{g}}}f\right)(x) = \int_G f(g) \rho_t^{\Delta_{\mathfrak{g}}}(g^{-1}x) dg. \quad (2.12)$$

All the Lie groups that appear in this paper (connected compact type groups and their complexifications) are unimodular, cf. Corollary 2.14, and so (2.12) characterizes the action of the heat operator for us.

The preceding properties of heat kernels hold generally, with no invariance assumption on the chosen inner product or topology of G . If the chosen inner product is $\text{Ad}(K)$ -invariant for some subgroup $K \subseteq G$, however, the resulting heat operator is right invariant over K , and the associated heat kernel is conjugation invariant by K .

Corollary 2.32. *Let $K \subseteq G$ be a compact type subgroup, and suppose that $\langle \cdot, \cdot \rangle_{\mathfrak{g}}$ is $\text{Ad}(K)$ -invariant. Then*

$$[e^{\frac{t}{2}\bar{\Delta}_{\mathfrak{g}}}, \hat{R}_k] = 0 \quad \text{and} \quad \rho_t^{\Delta_{\mathfrak{g}}}(kgk^{-1}) = \rho_t^{\Delta_{\mathfrak{g}}}(g)$$

for all $t > 0$, $k \in K$, and $g \in G$.

Proof. The corollary follows by combining the results in Lemma 2.24 with $V = \mathfrak{g}$, Remark 2.27, Theorem 2.30, and Proposition 2.17 in conjunction with Corollary 2.14. \square

If G is a Lie group and $K \subset G$ is a proper Lie subgroup with Lie algebra \mathfrak{k} , we will often wish to consider the Laplacian $\Delta_{\mathfrak{k}}$ as a differential operator on the full group G . The next proposition shows that, in so doing, its heat kernel $\rho_t^{\Delta_{\mathfrak{k}}}$ (intrinsic to K) is still the integral kernel of $e^{\frac{t}{2}\bar{\Delta}_{\mathfrak{k}}}$ acting on $L^2(G)$.

Proposition 2.33. *If K is a Lie subgroup of G and $\langle \cdot, \cdot \rangle_{\mathfrak{k}}$ is an inner product on $\mathfrak{k} = \text{Lie}(K)$, then as operators on $L^2(G)$ we have,*

$$e^{\frac{t}{2}\bar{\Delta}_{\mathfrak{k}}} = \int_K \rho_t^{\Delta_{\mathfrak{k}}}(k) \hat{R}_k dk,$$

where $\rho_t^{\Delta_{\mathfrak{k}}} : K \rightarrow (0, \infty)$ is the heat kernel on K given in Theorem 2.30 with G replaced by K everywhere.

Proof. Let $T_t := \int_K \rho_t^{\Delta_{\mathfrak{k}}}(k) \hat{R}_k dk$. It is straightforward to verify $\{T_t\}_{t>0}$ is a self-adjoint contraction semi-group on $L^2(G)$ and if $f \in C_c^\infty(G)$, then for each fixed $x \in G$,

$$\begin{aligned} \frac{d}{dt}(T_t f)(x) &= \frac{d}{dt} \int_K \rho_t^{\Delta_{\mathfrak{k}}}(k) f(xk) dk = \int_K \frac{d}{dt} \rho_t^{\Delta_{\mathfrak{k}}}(k) f(xk) dk \\ &= \int_K \frac{1}{2} \left(\Delta_{\mathfrak{k}} \rho_t^{\Delta_{\mathfrak{k}}} \right)(k) f(xk) dk \\ &= \int_K \rho_t^{\Delta_{\mathfrak{k}}}(k) \frac{1}{2} (\Delta_{\mathfrak{k}} f)(xk) dk = \frac{1}{2} T_t (\Delta_{\mathfrak{k}} f)(x). \end{aligned}$$

Hence it follows that

$$\frac{(T_t f)(x) - f(x)}{t} - \frac{1}{2} \Delta_{\mathfrak{k}} f(x) = \frac{1}{2t} \int_0^t [T_s (\Delta_{\mathfrak{k}} f)(x) - (\Delta_{\mathfrak{k}} f)(x)] ds$$

and therefore

$$\left\| \frac{T_t f - f}{t} - \frac{1}{2} \Delta_{\mathfrak{k}} f \right\|_{L^2(G)} \leq \frac{1}{2t} \int_0^t \|T_s(\Delta_{\mathfrak{k}} f) - \Delta_{\mathfrak{k}} f\|_{L^2(G)} ds \rightarrow 0 \text{ as } t \downarrow 0.$$

Hence, if A is the self-adjoint generator of the semigroup $\{T_t\}_{t>0}$ acting on $L^2(G)$, we have just shown $\frac{1}{2}\Delta_{\mathfrak{k}} \subset A$. Combined with Theorem 2.26, this implies that

$$\frac{1}{2}\bar{\Delta}_{\mathfrak{k}} \subset A = A^* \subset \left(\frac{1}{2}\Delta_{\mathfrak{k}}\right)^* = \frac{1}{2}\bar{\Delta}_{\mathfrak{k}}.$$

Thus $A = \frac{1}{2}\bar{\Delta}_{\mathfrak{k}}$, and consequently $T_t = e^{tA} = e^{t\frac{1}{2}\bar{\Delta}_{\mathfrak{k}}}$, completing the proof. \square

The next result, which is the final theorem of this section, regards the interaction of $\Delta_{\mathfrak{g}}$ and $\Delta_{\mathfrak{k}}$. In the presence of $\text{Ad}(K)$ -invariance, these two Laplacians commute, as do their closures and heat operators. The precise statement and proof follows the second author's paper [16, pp. 124-125].

Theorem 2.34. *Suppose that K is a compact type Lie subgroup of G , $\langle \cdot, \cdot \rangle_{\mathfrak{k}}$ is an inner product on $\mathfrak{k} = \text{Lie}(K)$, and $\langle \cdot, \cdot \rangle_{\mathfrak{g}}$ is an $\text{Ad}(K)$ -invariant inner product on \mathfrak{g} . Then*

- (1) $[e^{t\bar{\Delta}_{\mathfrak{k}}}, e^{s\bar{\Delta}_{\mathfrak{g}}}] = 0$ for all $s, t > 0$,
- (2) $e^{t\bar{\Delta}_{\mathfrak{k}}} e^{t\bar{\Delta}_{\mathfrak{g}}} = e^{t(\bar{\Delta}_{\mathfrak{k}} + \bar{\Delta}_{\mathfrak{g}})}$, and
- (3) the heat kernel $\rho_t^{(\Delta_{\mathfrak{k}} + \Delta_{\mathfrak{g}})}$ for $\Delta_{\mathfrak{k}} + \Delta_{\mathfrak{g}}$ may be expressed as

$$\rho_t^{(\Delta_{\mathfrak{k}} + \Delta_{\mathfrak{g}})}(x) = \int_K \rho_t^{\Delta_{\mathfrak{g}}}(xk^{-1}) \rho_t^{\Delta_{\mathfrak{k}}}(k) dk \quad (2.13)$$

$$= \int_K \rho_t^{\Delta_{\mathfrak{g}}}(k^{-1}x) \rho_t^{\Delta_{\mathfrak{k}}}(k) dk. \quad (2.14)$$

Proof. We take each item in turn.

- (1) By Corollary 2.32 (applied with \mathfrak{k} in place of \mathfrak{g}), $[e^{t\bar{\Delta}_{\mathfrak{k}}}, \hat{R}_k] = 0$ for all $k \in K$, and hence using Proposition 2.33,

$$[e^{t\bar{\Delta}_{\mathfrak{k}}}, e^{\frac{s}{2}\bar{\Delta}_{\mathfrak{g}}}] = \left[e^{t\bar{\Delta}_{\mathfrak{k}}}, \int_K \rho_s^{\mathfrak{k}}(k) \hat{R}_k dk \right] = \int_K \rho_s^{\mathfrak{k}}(k) [e^{t\bar{\Delta}_{\mathfrak{k}}}, \hat{R}_k] dk = 0.$$

- (2) Since $\{e^{t\bar{\Delta}_{\mathfrak{k}}}\}_{t>0}$ and $\{e^{t\bar{\Delta}_{\mathfrak{g}}}\}_{t>0}$ are two commuting self-adjoint contraction semi-groups it follows that $T_t := e^{t\bar{\Delta}_{\mathfrak{k}}} e^{t\bar{\Delta}_{\mathfrak{g}}}$ for $t > 0$ is also a self-adjoint contraction semi-group. Moreover if $f \in \mathcal{D}(\Delta_{\mathfrak{k}}) \cap \mathcal{D}(\Delta_{\mathfrak{g}})$, then

$$\frac{T_t f - f}{t} = e^{t\bar{\Delta}_{\mathfrak{g}}} \frac{e^{t\bar{\Delta}_{\mathfrak{k}}} f - f}{t} + \frac{e^{t\bar{\Delta}_{\mathfrak{g}}} f - f}{t} \rightarrow \bar{\Delta}_{\mathfrak{k}} f + \bar{\Delta}_{\mathfrak{g}} f \text{ as } t \downarrow 0.$$

It follows $\Delta_{\mathfrak{k}} + \Delta_{\mathfrak{g}} \subset \bar{\Delta}_{\mathfrak{k}} + \bar{\Delta}_{\mathfrak{g}} \subset A$, where A is the generator of $\{T_t\}_{t>0}$. Hence it follows again from Theorem 2.26 (as in the proof of Proposition 2.33) that $\bar{\Delta}_{\mathfrak{k}} + \bar{\Delta}_{\mathfrak{g}} = A$. Item (2) follows.

- (3) Replacing t by $t/2$, Lemma 2.21, Theorem 2.30, and Proposition 2.33 show that the statement of item (2) is equivalent to

$$\Gamma_{\rho_t^{(\Delta_{\mathfrak{k}} + \Delta_{\mathfrak{g}})} \lambda_G} = \Gamma_{\rho_t^{\Delta_{\mathfrak{g}}} \lambda_G} \Gamma_{\rho_t^{\Delta_{\mathfrak{k}}} \lambda_K} = \Gamma_{(\rho_t^{\Delta_{\mathfrak{g}}} \lambda_G) * (\rho_t^{\Delta_{\mathfrak{k}}} \lambda_K)} \quad (2.15)$$

where λ_G and λ_K are right invariant Haar measures on G and K respectively. The identity (2.15) is, in turn, equivalent to (2.13) since, for all $f \in C_c(G)$ we have

$$\begin{aligned} \int f d((\rho_t^{\Delta_{\mathfrak{g}}} \lambda_G) * (\rho_t^{\Delta_{\mathfrak{k}}} \lambda_K)) &= \int_{G \times K} f(xk) \rho_t^{\Delta_{\mathfrak{g}}}(x) \rho_t^{\Delta_{\mathfrak{k}}}(k) dx dk \\ &= \int_{G \times K} f(x) \rho_t^{\Delta_{\mathfrak{g}}}(xk^{-1}) \rho_t^{\Delta_{\mathfrak{k}}}(k) dx dk \\ &= \int_{G \times K} f(x) \left[\int_K \rho_t^{\Delta_{\mathfrak{g}}}(xk^{-1}) \rho_t^{\Delta_{\mathfrak{k}}}(k) dk \right] dx. \end{aligned}$$

Equation (2.14) directly follows from (2.13) and Corollary 2.32. \square

Remark 2.35. By differentiating Theorem 2.34(1) with respect to s, t at 0, we deduce that $\bar{\Delta}_{\mathfrak{k}}$ and $\bar{\Delta}_{\mathfrak{g}}$ commute. It is possible to derive this (pre-closure) using more elementary methods, at least in the case that the inner product on \mathfrak{g} restricts to the inner product on \mathfrak{k} ; however, the authors do not see an easy route from this commutation to prove the commutation of the heat operators directly.

2.7. The Heat Operator on Matrix Entries. In case $G = \mathbb{R}^k$, it is very convenient to do computations with the heat operator on polynomial functions. Although these functions are not in $L^2(\mathbb{R}^k)$, since the Laplacian is a nilpotent operator, one can naively make sense of $e^{\frac{t}{2}\Delta_{\mathbb{R}^k}} f$ as a (finitely-terminating) power series for any polynomial f . It is then an easy matter to verify that the integral formula (2.12) for the heat operator coincides with this series: if f is a polynomial on \mathbb{R}^k , then

$$\int_{\mathbb{R}^k} f(y) \rho_t^{\Delta_{\mathbb{R}^k}}(x - y) dy = \sum_{n=0}^{\infty} \frac{(t/2)^n}{n!} (\Delta_{\mathbb{R}^k})^n f(x) \quad (2.16)$$

The series is actually finite, and (2.16) is easy to prove directly. It also follows from the more general Lemma 2.44 below.

For many computations, we will need the analog of polynomial functions on a general Lie group; these are *matrix entries*, which we define as follows.

Definition 2.36. Let G be a Lie group. Let (π, V_{π}) be a finite dimensional complex representation of G , and let $A \in \text{End}(V_{\pi})$ be a fixed endomorphism. The associated **matrix entry** function $f_{\pi, A}$ on G is the function

$$f_{\pi, A}(x) = \text{Tr}(\pi(x)A).$$

In the case $A = I$ is the identity, the matrix entry $f_{\pi, I} = \chi_{\pi}$ is called the **character** of π .

Remark 2.37. Some authors restrict the term *matrix entries* to $f_{\pi, A}$ with π irreducible; in that case, what we call matrix entries above would be linear combinations of matrix entries, as the next lemma shows.

Lemma 2.38. For any Lie group G , the set of matrix entries on G forms a self-adjoint complex algebra.

Proof. It is straightforward to compute that, for $\lambda \in \mathbb{C}$, $\lambda f_{\pi, A} = f_{\pi, \lambda A}$, while sums and products satisfy $f_{\pi, A} + f_{\pi, B} = f_{\pi, A+B}$ and $f_{\pi, A} f_{\pi, B} = f_{\pi \otimes \sigma, A \otimes B}$. For complex conjugation, we must define the complex conjugate of a representation and an endomorphism. This can be done invariantly, but for our purposes there is no reason not to simply choose

a basis. Given a representation (π, V_π) of dimension d , choose a complex linear isomorphism $\varphi: V_\pi \rightarrow \mathbb{C}^d$, and let $[\pi(x)] = \varphi \circ \pi(x) \circ \varphi^{-1}$ and $[A] = \varphi \circ A \circ \varphi^{-1}$. As $d \times d$ complex matrices, both $[\pi(x)]$ and $[A]$ have complex conjugates $\overline{[\pi(x)]}$ and $\overline{[A]}$, defined entry-wise. Then

$$\bar{f}_{\pi,A}(x) = \overline{\text{Tr}(\pi(x)A)} = \overline{\text{Tr}([\pi(x)][A])} = \text{Tr}(\overline{[\pi(x)]} \overline{[A]}). \quad (2.17)$$

The map $\overline{[\pi]}: G \rightarrow \text{GL}(\mathbb{C}^d)$ given by $\overline{[\pi]}(x) = \overline{[\pi(x)]}$ is a representation of G on \mathbb{C}^d , and (2.17) shows that

$$\bar{f}_{\pi,A} = f_{\overline{[\pi]}, \overline{A}}$$

is also a matrix entry of G . This concludes the proof. \square

The algebra of matrix entries is also closed under the action of left-invariant differential operators on G .

Lemma 2.39. *Let $\{X_1, \dots, X_d\}$ be a basis of \mathfrak{g} , let P be a noncommutative polynomial in d variables, and let $\mathcal{L} = P(\tilde{X}_1, \dots, \tilde{X}_d)$ be a left invariant differential operator. Given a representation (π, V_π) of G , define $L_\pi = P(\pi_*(X_1), \dots, \pi_*(X_d)) \in \text{End}(V_\pi)$. Then for any endomorphism A of V_π ,*

$$\mathcal{L} f_{\pi,A} = f_{\pi, L_\pi \cdot A}.$$

Proof. Given any $X \in \mathfrak{g}$ and $x \in G$, we compute

$$\begin{aligned} \tilde{X} f_{\pi,A}(x) &= \left. \frac{d}{dt} \right|_{t=0} f_{\pi,A}(x e^{tX}) = \left. \frac{d}{dt} \right|_{t=0} \text{Tr}(\pi(x) \pi(e^{tX}) A) = \text{Tr}(\pi(x) \pi_*(X) A) \\ &= f_{\pi, \pi_*(X) \cdot A}(x). \end{aligned}$$

The result now follows for monomials P by induction, and then in general by linearity. \square

Matrix entries are smooth functions on G . If $G = \mathbb{R}^k$, all polynomials are matrix entries, as are exponential functions (e.g. $\pi: \mathbb{R} \rightarrow \text{GL}(\mathbb{C})$ given by $\pi(x) = e^x$ is a group homomorphism). In general, matrix can have exponential growth, but they cannot grow super-exponentially. This follows from the following lemma.

Lemma 2.40. *Let $\pi: G \rightarrow \text{End}(V)$ be a finite dimensional representation of G , let $\|\cdot\|$ be a norm on V , and let $\|\cdot\|_{\text{op}}$ denote the operator norm it induces on $\text{End}(V)$. Then there exists $\beta < \infty$ such that*

$$\|\pi(x)\|_{\text{op}} \leq e^{\beta|x|} \text{ for all } x \in G. \quad (2.18)$$

(Here $|x|$ denotes the Riemannian distance from e to x , cf. Notation 2.29.)

Proof. Let $x \in G$ and $g: [0, 1] \rightarrow G$ be any piecewise C^1 path in G such that $g(0) = e$ and $g(1) = x$, and further let $\dot{A}(t) := L_{g(t)^{-1}} \dot{g}(t)$. Then

$$\frac{d}{dt} \pi(g(t)) = \pi(g(t)) \pi_*(\dot{A}(t)) \quad \text{with} \quad \pi(g(0)) = I.$$

The usual Gronwall estimates then show that

$$\begin{aligned} \|\pi(x)\|_{\text{op}} &= \|\pi(g(1))\|_{\text{op}} \leq \exp \left(\int_0^1 \|d\pi(\dot{A}(t))\|_{\text{op}} dt \right) \\ &\leq \exp \left(\beta \int_0^1 \|\dot{A}(t)\|_{\mathfrak{g}} dt \right) = \exp(\beta \ell(g)) \end{aligned}$$

where $\ell(g)$ is the Riemannian length of the path g and $\beta > 0$ is chosen so that $\|\pi_*(A)\|_{\text{op}} \leq \beta \|A\|_{\mathfrak{g}}$ for all $A \in \mathfrak{g}$. Minimizing the right side over all such paths g joining e to x proves (2.18). \square

Corollary 2.41. *All matrix entries $f_{\pi,A}$ grow at most exponentially, and are in $L^p(G, \rho_t^{\Delta_{\mathfrak{g}}})$ for all $p < \infty$ with respect to any left invariant Riemannian metric.*

Proof. Choosing an inner product on V_{π} , equip $\text{End}(V_{\pi})$ with its Schatten norms: for $1 \leq p < \infty$,

$$\|A\|_p := \left(\text{Tr}[(AA^*)^{p/2}] \right)^{1/p}.$$

As $p \rightarrow \infty$, $\|A\|_p \rightarrow \|A\|_{\text{op}}$. These norms satisfy Hölder’s inequality; in particular $|\text{Tr}(AB)| \leq \|A\|_1 \|B\|_{\text{op}}$, cf. [29]. Now, following Lemma 2.40, we have

$$|f_{\pi,A}(x)| = |\text{Tr}(\pi(x)A)| \leq \|A\|_1 \|\pi(x)\|_{\text{op}} \leq \|A\|_1 e^{\beta|x|}. \quad (2.19)$$

This proves exponential growth. The statement about containment in L^p of any heat kernel measure now follows from Theorem 2.30(7). \square

Because of this exponential growth, we can now naïvely define the exponential $e^{\mathcal{L}}$ of any left invariant differential operator (given as in Lemma 2.39) acting on matrix entries, by the power series:

$$e^{\mathcal{L}} f_{\pi,A} := \sum_{n=0}^{\infty} \frac{1}{n!} \mathcal{L}^n f_{\pi,A} = \sum_{n=0}^{\infty} \frac{1}{n!} f_{\pi, (L_{\pi})^n \cdot A} = f_{\pi, e^{L_{\pi}} \cdot A}. \quad (2.20)$$

Indeed, the series in the penultimate equality converges since, from (2.19), we have

$$|\mathcal{L}^n f_{\pi,A}(x)| \leq \|A\|_1 (\|L_{\pi}\|_1)^n e^{\beta|x|} \quad (2.21)$$

where β is determined by π , and we have used the fact that the Schatten norms are submultiplicative. Then the last equality holds by Lemma 2.39 and linearity and continuity.

In particular, we have a putative “heat operator” $e^{\frac{t}{2}\Delta_{\mathfrak{g}}}$ acting on matrix entries $f_{\pi,A}$ by

$$e^{\frac{t}{2}\Delta_{\mathfrak{g}}} f_{\pi,A} = f_{\pi, e^{tC_{\pi}/2} A} \quad (2.22)$$

where

$$C_{\pi} = \sum_{j=1}^d \pi_*(X_j)^2. \quad (2.23)$$

Remark 2.42. The endomorphism C_{π} is sometimes called the *Casimir invariant* of π . It commutes with all endomorphisms in the image of π_* ; this follows from Lemma 2.24 applied to the function $f = \pi$.

We should not confuse (2.22) with the heat operator $e^{\frac{t}{2}\tilde{\Delta}_{\mathfrak{g}}}$ of Theorem 2.26: if the Lie group G is not compact, then the matrix entry functions $f_{\pi,A}$ are not in $L^2(G)$, which is where $e^{\frac{t}{2}\tilde{\Delta}_{\mathfrak{g}}}$ is defined. Nevertheless, the two do coincide in the appropriate sense: both are given by convolution with the heat kernel $\rho_t^{\Delta_{\mathfrak{g}}}$, cf. (2.12). To prove this, we first need the following notation and lemma.

Notation 2.43. *Let G be a Lie group with Lie algebra \mathfrak{g} , with a specified inner product. For $f \in C^1(G)$, the gradient of f , $\nabla_{\mathfrak{g}} f$, is the \mathfrak{g} -valued continuous function determined by*

$$\langle \nabla_{\mathfrak{g}} f(x), V \rangle_{\mathfrak{g}} = \tilde{V} f(x), \quad \forall V \in \mathfrak{g}, x \in G.$$

If $\{X_1, \dots, X_d\}$ is any orthonormal basis for \mathfrak{g} , then

$$\nabla_{\mathfrak{g}} f = \sum_{j=1}^d (\tilde{X}_j f) X_j. \quad (2.24)$$

The gradient will appear in the “product rule” for the Laplacian in the proof of the following lemma.

Lemma 2.44. *If $f \in C^2(G, \mathbb{C})$ is a function such that f , $\nabla_{\mathfrak{g}} f$, and $\Delta_{\mathfrak{g}} f$ have at most exponential growth, then*

$$\frac{d}{dt} \int_G f(x) \rho_t^{\Delta_{\mathfrak{g}}}(x) dx = \frac{1}{2} \int_G (\Delta_{\mathfrak{g}} f)(x) \rho_t^{\Delta_{\mathfrak{g}}}(x) dx \quad \text{for all } t > 0. \quad (2.25)$$

Proof. The proof of this result follows the same lines of [6, Lemma 3.8] so we will only briefly sketch the argument here. The key to the proof is the existence of cutoff functions, $\{h_n\}_{n=1}^{\infty} \subset C_c^{\infty}(G, [0, 1])$ such that $h_n(x) = 1$ if $|x| \leq n$ and $\sup_n \sup_{x \in G} |\mathcal{L} h_n(x)| < \infty$ for every left invariant linear differential operator \mathcal{L} on G , see [6, Lemma 3.6].

For $n \in \mathbb{N}$, let

$$F_n(t) := \int_G f(x) \rho_t^{\Delta_{\mathfrak{g}}}(x) h_n(x) dx$$

so that

$$\begin{aligned} \dot{F}_n(t) &= \int_G \frac{1}{2} (\Delta_{\mathfrak{g}} \rho_t^{\Delta_{\mathfrak{g}}})(x) f(x) h_n(x) dx \\ &= \frac{1}{2} \int_G \rho_t^{\Delta_{\mathfrak{g}}}(x) (\Delta_{\mathfrak{g}} [f h_n])(x) dx \\ &= \frac{1}{2} \int_G \rho_t^{\Delta_{\mathfrak{g}}}(x) (\Delta_{\mathfrak{g}} f \cdot h_n + 2 \nabla_{\mathfrak{g}} f \cdot \nabla_{\mathfrak{g}} h_n + f \Delta_{\mathfrak{g}} h_n)(x) dx \end{aligned}$$

where $(\nabla_{\mathfrak{g}} f \cdot \nabla_{\mathfrak{g}} h)(x) = \langle \nabla_{\mathfrak{g}} f(x), \nabla_{\mathfrak{g}} h(x) \rangle_{\mathfrak{g}}$. Using Theorem 2.30(7) and the properties of the cutoff functions $\{h_n\}_{n=1}^{\infty}$, it is easy to show (by the dominated convergence theorem) that, as $n \rightarrow \infty$,

$$\begin{aligned} F_n(t) &\rightarrow \int_G f(x) \rho_t^{\Delta_{\mathfrak{g}}}(x) dx \quad \text{and} \\ \dot{F}_n(t) &\rightarrow \frac{1}{2} \int_G \rho_t^{\Delta_{\mathfrak{g}}}(x) \Delta_{\mathfrak{g}} f(x) dx \end{aligned}$$

uniformly for t in compact subsets of $(0, \infty)$. This suffices to verify (2.25). \square

This brings us to the main result of this section, generalizing (2.16) to general Lie groups.

Proposition 2.45. *Let G be a Lie group, with Lie algebra \mathfrak{g} possessing a specified inner product, and let $\rho_t^{\Delta_{\mathfrak{g}}}$ denote the heat kernel of Theorem 2.30. Then for any matrix entry function $f_{\pi, A}$ on G ,*

$$\int_G f_{\pi, A}(g) \rho_t^{\Delta_{\mathfrak{g}}}(g^{-1}x) dg = \left(e^{\frac{t}{2} \Delta_{\mathfrak{g}}} f_{\pi, A} \right)(x) = f_{\pi, e^{tC_{\pi}/2} A}(x) \quad (2.26)$$

cf. (2.22). In particular, the integral of $f_{\pi, A}$ against the heat kernel can be computed as

$$\int_G f_{\pi, A}(g) \rho_t^{\Delta_{\mathfrak{g}}}(g) dg = \left(e^{\frac{t}{2} \Delta_{\mathfrak{g}}} f_{\pi, A} \right)(e) = f_{\pi, e^{tC_{\pi}/2} A}(e). \quad (2.27)$$

Proof. It actually suffices to prove (2.27). Indeed, using Theorem 2.30(1), we can change variables to see that

$$\int_G f_{\pi,A}(g) \rho_t^{\Delta_{\mathfrak{g}}}(g^{-1}x) dg = \int_G f_{\pi,A}(xg) \rho_t^{\Delta_{\mathfrak{g}}}(g) dg$$

and since $f_{\pi,A}(xg) = \text{Tr}(\pi(xg)A) = \text{Tr}(\pi(x)\pi(g)A) = \text{Tr}(\pi(g) \cdot A\pi(x)) = f_{\pi,A\pi(x)}(g)$, (2.27) applied to $f_{\pi,A\pi(x)}$ yields (2.26).

Define $F(t)$ to be the left-hand-side of (2.27). By Lemma 2.39, $\nabla_{\mathfrak{g}} f_{\pi,A}$ and $\Delta_{\mathfrak{g}} f_{\pi,A}$ are matrix entries (\mathfrak{g} -valued in the case of the gradient), and hence by repeated application of Corollary 2.41 and Lemma 2.44, we find that for any $n \in \mathbb{N}$,

$$F^{(n)}(t) = \int_G (\Delta_{\mathfrak{g}}/2)^n f_{\pi,A}(g) \rho_t^{\Delta_{\mathfrak{g}}}(g) dg = \frac{1}{2^n} \int_G \rho_t^{\Delta_{\mathfrak{g}}}(g) f_{\pi,(C_{\pi})^n A}(g) dg.$$

Applying Taylor's theorem, using (2.21) to estimate the remainder, we find that for any $t_0 > 0$,

$$\begin{aligned} F(t) &= \sum_{n=0}^{\infty} \frac{1}{n!} F^{(n)}(t_0) (t - t_0)^n \\ &= \int_G \rho_{t_0}^{\Delta_{\mathfrak{g}}}(g) \sum_{n=0}^{\infty} \frac{(t - t_0)^n}{2^n n!} f_{\pi,(C_{\pi})^n A}(g) dg \\ &= \int_G \rho_{t_0}^{\Delta_{\mathfrak{g}}}(g) f_{\pi,e^{(t-t_0)C_{\pi}/2}A}(g) dg. \end{aligned}$$

Letting $t_0 \downarrow 0$, using a simple cutoff argument along with Theorem 2.30(8), completes the proof of (2.27). \square

Comparing (2.12) and (2.26), we see that the two definitions of the heat operator (via spectral theory on $L^2(G)$ with $C_c^\infty(G)$ as a core, vs. as a power series acting on matrix entries) are consistent, even though they are defined on spaces that may intersect only at 0.

2.8. An Averaging Theorem. In this section, we prove a regularity property of heat kernels on G associated to $\text{Ad}(K)$ -invariant Laplacians, for $K \subseteq G$ compact. The following theorem was essentially proved in [16], in a context that applied to the case $G = K_{\mathbb{C}}$. We give a proof here that shows explicitly that the result (which is of independent interest) does not depend on any complex structure on G , or any special relationship between K and G .

To begin, we need the following lemma.

Lemma 2.46. *Let G be a Lie group with Lie algebra \mathfrak{g} , and let $K \subseteq G$ be a compact type Lie subgroup with Lie algebra $\mathfrak{k} \subseteq \mathfrak{g}$. Fix an $\text{Ad}(K)$ -invariant inner product on \mathfrak{g} , and denote by $\mathfrak{k}^\perp \subseteq \mathfrak{g}$ the orthogonal complement of \mathfrak{k} . Then both \mathfrak{k} and \mathfrak{k}^\perp are invariant subspaces for Ad_k for each $k \in K$.*

Proof. Since \mathfrak{k} is the Lie algebra of K , it is automatically invariant under Ad_k for each $k \in K$ (since K is invariant under conjugation by $k \in K$). Now, let Ad_k^* denote the adjoint of the operator Ad_k with respect to the given inner product. For $X \in \mathfrak{k}$ and $Y \in \mathfrak{k}^\perp$,

$$\langle \text{Ad}_k^*(Y), X \rangle = \langle Y, \text{Ad}_k(X) \rangle = 0$$

since $\text{Ad}_k(X) \in \mathfrak{k}$. This shows that $\text{Ad}_k(Y) \in \mathfrak{k}^\perp$, so \mathfrak{k}^\perp is invariant under Ad_k^* for each $k \in K$. Since the inner product is $\text{Ad}(K)$ -invariant, Ad_k is unitary, and so $\text{Ad}_k^* = \text{Ad}_k^{-1} = \text{Ad}_{k^{-1}}$. As this holds for all $k \in K$, it follows that \mathfrak{k}^\perp is invariant under Ad_k for each $k \in K$, as desired. \square

Remark 2.47. Typically, the complement of a Lie subalgebra is not a Lie subalgebra, and is therefore not generally closed under the adjoint map. This property of compact type subgroups goes a long way in explaining their nice structure.

Theorem 2.48. *Let G be a Lie group with Lie algebra \mathfrak{g} , and let $K \subseteq G$ be a compact Lie subgroup. Fix an $\text{Ad}(K)$ -invariant inner product on \mathfrak{g} , let $\Delta_{\mathfrak{g}}$ be the associated Laplacian, and let $\rho_t^{\Delta_{\mathfrak{g}}}$ be the associated heat kernel. For each $t > 0$, there is a constant $C(t) > 0$ (depending on the given inner product) such that, for all $k \in K$, and $x \in G$,*

$$C(t)^{-1} \rho_t^{\Delta_{\mathfrak{g}}}(xk) \leq \rho_t^{\Delta_{\mathfrak{g}}}(x) \leq C(t) \rho_t^{\Delta_{\mathfrak{g}}}(xk).$$

In other words: the heat kernel measure $\rho_t^{\Delta_{\mathfrak{g}}}(x) dx$ is (uniformly) quasi-invariant under right multiplication by K .

Proof. Let $\mathfrak{k} \subseteq \mathfrak{g}$ denote the Lie algebra of K . Denote by $\langle \cdot, \cdot \rangle_{\mathfrak{g}}$ the given $\text{Ad}(K)$ -invariant inner product on \mathfrak{g} . Denote the dimensions of G and K as d_G and d_K respectively. Fix an orthonormal basis $\{X_1, \dots, X_{d_G}\}$ for \mathfrak{g} with the property that $\{X_1, \dots, X_{d_K}\}$ is an orthonormal basis for \mathfrak{k} . Define four operators

$$\Delta_{\mathfrak{k}} := \sum_{j=1}^{d_K} \tilde{X}_j^2, \quad \Delta_{\mathfrak{k}^\perp} := \sum_{j=d_K+1}^{d_G} \tilde{X}_j^2, \quad \Delta_{\mathfrak{g}} := \Delta_{\mathfrak{k}} + \Delta_{\mathfrak{k}^\perp} \\ \Delta'_{\mathfrak{g}} := \frac{1}{2} \Delta_{\mathfrak{k}} + \Delta_{\mathfrak{k}^\perp}.$$

By Theorem 2.26, all four operators are essentially self-adjoint in $L^2(G)$, with $C_c^\infty(G)$ as a common core. The operator $\Delta_{\mathfrak{g}}$ is the Laplacian on G determined by $\langle \cdot, \cdot \rangle_{\mathfrak{g}}$. Also notice that $\Delta_{\mathfrak{g}} = \frac{1}{2} \Delta_{\mathfrak{k}} + \Delta'_{\mathfrak{g}}$, and that $\Delta'_{\mathfrak{g}}$ is the Laplacian associated to the modified inner product $\langle \cdot, \cdot \rangle'_{\mathfrak{g}}$ defined as follows: for $X_1, X_2 \in \mathfrak{k}$ and $Y_1, Y_2 \in \mathfrak{k}^\perp$,

$$\langle X_1 + Y_1, X_2 + Y_2 \rangle'_{\mathfrak{g}} := \frac{1}{2} \langle X_1, X_2 \rangle_{\mathfrak{g}} + \langle Y_1, Y_2 \rangle_{\mathfrak{g}}.$$

Using Lemma 2.46, it is straightforward to verify that $\langle \cdot, \cdot \rangle'_{\mathfrak{g}}$ is also $\text{Ad}(K)$ -invariant. Its restriction to \mathfrak{k} is just $\frac{1}{2}$ times the original inner product, and $\frac{1}{2} \Delta_{\mathfrak{k}}$ is the Laplacian associated to this scaled inner product on \mathfrak{k} (considered here as an operator on G , cf. Proposition 2.33).

Thus, we may apply Theorem 2.34 in this context, and we find that

$$e^{t\bar{\Delta}_{\mathfrak{g}}} = e^{t\left(\frac{1}{2}\bar{\Delta}_{\mathfrak{k}} + \bar{\Delta}'_{\mathfrak{g}}\right)} = e^{\frac{t}{2}\bar{\Delta}_{\mathfrak{k}}} e^{t\bar{\Delta}'_{\mathfrak{g}}}$$

and, moreover,

$$\rho_t^{\Delta_{\mathfrak{g}}}(x) = \int_K \rho_t^{\Delta'_{\mathfrak{g}}}(xk^{-1}) \rho_t^{\frac{1}{2}\Delta_{\mathfrak{k}}}(k) dk. \quad (2.28)$$

Now, the heat kernel $\rho_t^{\frac{1}{2}\Delta_{\mathfrak{k}}}$ is a continuous positive function on the compact group K , and hence there are constants $0 < \alpha(t) \leq \beta(t) < \infty$ such that $\alpha(t) \leq \rho_t^{\frac{1}{2}\Delta_{\mathfrak{k}}}(k) \leq \beta(t)$ for all $k \in K$. Thus, following (2.28), we have, for any $x \in G$,

$$\rho_t^{\Delta_{\mathfrak{g}}}(x) = \int_K \rho_t^{\Delta'_{\mathfrak{g}}}(xk^{-1}) \rho_t^{\frac{1}{2}\Delta_{\mathfrak{k}}}(k) dk \geq \alpha(t) \int_K \rho_t^{\Delta'_{\mathfrak{g}}}(xk^{-1}) dk. \quad (2.29)$$

At the same time, for $k \in K$, again by (2.28) we have

$$\rho_t^{\Delta_{\mathfrak{g}}}(xk) = \int_K \rho_t^{\Delta'_{\mathfrak{g}}}(xya^{-1}) \rho_t^{\frac{1}{2}\Delta_{\mathfrak{k}}}(a) da \leq \beta(t) \int_K \rho_t^{\Delta'_{\mathfrak{g}}}(xka^{-1}) da.$$

In the last integral, we make the substitution $\ell = ka^{-1}$; then $d\ell = dk$ (since dk is the right Haar measure). Combining this with (2.29), we find

$$\rho_t^{\Delta_{\mathfrak{g}}}(xk) \leq \beta(t) \int_K \rho_t^{\Delta'_{\mathfrak{g}}}(x\ell^{-1}) d\ell \leq \frac{\beta(t)}{\alpha(t)} \rho_t^{\Delta_{\mathfrak{g}}}(x).$$

Hence, we can take $C(t) = \frac{\beta(t)}{\alpha(t)}$, verifying the first inequality. An analogous argument shows that the second inequality holds as well. \square

We refer to Theorem 2.48 as an *averaging* theorem because we will apply it in the following form.

Corollary 2.49. *Let G , K , $\rho_t^{\Delta_{\mathfrak{g}}}$, and $C(t)$ be as in Theorem 2.48. Let ν be any probability measure on K , and define the ν -average heat kernel ν_t on G as*

$$\nu_t(x) = \int_K \rho_t^{\Delta_{\mathfrak{g}}}(xk) \nu(dk).$$

Then

$$C(t)^{-1} \nu_t(x) \leq \rho_t^{\Delta_{\mathfrak{g}}}(x) \leq C(t) \nu_t(x), \quad \text{for all } x \in G.$$

The proof of Corollary 2.49 is an immediate consequence of Theorem 2.48. We will apply it (with ν taken to be the Haar measure) to determine the range of the complex time Segal–Bargmann transform in Section 3.3, and this averaged heat kernel will also play a role in Section 4.2.4.

3. INVARIANT METRICS AND MEASURES ON $K_{\mathbb{C}}$

If G is a Lie group and $K \subseteq G$ is a compact Lie subgroup with Lie algebra \mathfrak{k} , then one can produce an $\text{Ad}(K)$ -invariant inner product on $\text{Lie}(G)$ by averaging any inner product over the Haar measure on K , as above. This begs the question: how many genuinely distinct $\text{Ad}(K)$ -invariant inner products does G possess? To make this question meaningful, we first fix an Ad -invariant inner product $\langle \cdot, \cdot \rangle_{\mathfrak{k}}$ on K ; then we can ask about the space of $\text{Ad}(K)$ -invariant inner product on G that extend $\langle \cdot, \cdot \rangle_{\mathfrak{k}}$. A partial answer to this question (in the slightly more general case that K is compact type) is given below, in the case that $G = K_{\mathbb{C}}$ is the complexification of K , cf. Section 2.1.

3.1. Invariant Inner Products and Laplacians on $K_{\mathbb{C}}$. Fix a compact type Lie group K , and an $\text{Ad}(K)$ -invariant inner product $\langle \cdot, \cdot \rangle_{\mathfrak{k}}$ on its Lie algebra \mathfrak{k} . Let $K_{\mathbb{C}}$ denote the complexification of K (cf. Section 2.1); in particular $\mathfrak{k}_{\mathbb{C}} \equiv \text{Lie}(K_{\mathbb{C}}) = \mathfrak{k} \oplus J\mathfrak{k}$. So every vector $Z \in \mathfrak{k}_{\mathbb{C}}$ has a unique decomposition $Z = V + JW$ for $V, W \in \mathfrak{k}$ (cf. Section 2.1).

Consider the following three-parameter family of inner products on $K_{\mathbb{C}}$.

$$\langle V_1 + JW_1, V_2 + JW_2 \rangle_{a,b,c} := a\langle V_1, V_2 \rangle_{\mathfrak{k}} + b\langle W_1, W_2 \rangle_{\mathfrak{k}} + c(\langle V_1, W_2 \rangle_{\mathfrak{k}} + \langle V_2, W_1 \rangle_{\mathfrak{k}}) \quad (3.1)$$

for $V_1, V_2, W_1, W_2 \in \mathfrak{k}$, where $a, b > 0$ and $c^2 < ab$. It is straightforward to verify that the symmetric bilinear forms in (3.1) are real inner products on $\mathfrak{k}_{\mathbb{C}}$ (precisely under the conditions on a, b, c stated below the equation), and are all $\text{Ad}(K)$ -invariant. The main theorem of this section is that, in the case that K is simple, this is a complete characterization of all $\text{Ad}(K)$ -invariant inner products on $K_{\mathbb{C}}$.

Theorem 3.1. *If K is a simple, compact type Lie group, then \mathfrak{k} has a unique (up to scale) Ad -invariant real inner product $\langle \cdot, \cdot \rangle_{\mathfrak{k}}$, and all $\text{Ad}(K)$ -invariant real inner products on $\mathfrak{k}_{\mathbb{C}}$ have the form (3.1).*

Remark 3.2. For example, $K = \mathrm{SU}(n)$ is simple, with complexification $K_{\mathbb{C}} = \mathrm{SL}(n, \mathbb{C})$. Hence (3.1) characterizes all $\mathrm{Ad}(\mathrm{SU}(n))$ -invariant inner products on $\mathrm{SL}(n, \mathbb{C})$, where $\langle X, Y \rangle_{\mathrm{su}(n)} = \mathrm{Tr}(XY^*)$ is the unique (up to scale) Ad -invariant inner product on $\mathrm{su}(n)$. In that case, the family can be written explicitly in terms of the trace as

$$\langle A, B \rangle_{a,b,c} = \frac{1}{2}(a+b)\Re\mathrm{Tr}(AB^*) + \frac{1}{2}\Re[(a-b-2ic)\mathrm{Tr}(AB)]. \quad (3.2)$$

Extending to $\mathrm{U}(n)$ and its complexification $\mathrm{GL}(n, \mathbb{C})$, it is easy to compute that all $\mathrm{Ad}(\mathrm{U}(n))$ -invariant inner products are of the form (3.1) plus one more term, involving the 1-dimensional subspace spanned by the identity matrix; extending (3.2), there is one more term involving $\mathrm{Tr}(A)\mathrm{Tr}(B)$. In [3, 20, 21], the third author studied the large- n limits of the diffusion processes on $\mathrm{GL}(n, \mathbb{C})$ invariant with respect to the inner products $\langle \cdot, \cdot \rangle_{a,b,0}$. Part of the motivation for the present work was the question of whether those were the largest class of appropriately invariant diffusions; the answer provided by Theorem 3.1 is no, and the associated diffusions will be explored in a future publication.

Proof. Let $\langle \cdot, \cdot \rangle$ be an $\mathrm{Ad}(K)$ -invariant inner product on $\mathfrak{k}_{\mathbb{C}}$, and let $\langle \cdot, \cdot \rangle_{\mathfrak{k}}$ be any Ad -invariant inner product on \mathfrak{k} . We will make use of $\langle \cdot, \cdot \rangle_{1,1,0}$ from (3.1) as a reference inner product; that is

$$\langle V_1 + JW_1, V_2 + JW_2 \rangle_{1,0,0} = \langle V_1, V_2 \rangle_{\mathfrak{k}} + \langle W_1, W_2 \rangle_{\mathfrak{k}}. \quad (3.3)$$

Since $\langle \cdot, \cdot \rangle$ and $\langle \cdot, \cdot \rangle_{1,1,0}$ are both inner products on the finite dimensional (real) vector space $\mathfrak{k}_{\mathbb{C}}$, there is a real-linear endomorphism $M: \mathfrak{k}_{\mathbb{C}} \rightarrow \mathfrak{k}_{\mathbb{C}}$ such that

$$\langle Z_1, Z_2 \rangle = \langle Z_1, M(Z_2) \rangle_{1,1,0}, \quad \forall Z_1, Z_2 \in \mathfrak{k}_{\mathbb{C}}.$$

Using the decomposition $\mathfrak{k}_{\mathbb{C}} = \mathfrak{k} \oplus J\mathfrak{k}$ and the fact that J is real linear, we decompose M in terms of four endomorphisms $A, B, C_1, C_2 \in \mathrm{End}(\mathfrak{k})$ as follows: $M(V + JW) = M(V) + M(JW)$, where $M(V) = A(V) + JC_1(V)$ and $M(JW) = C_2(W) + JB(W)$. Thus

$$\begin{aligned} & \langle V_1 + JW_1, V_2 + JW_2 \rangle \\ &= \langle V_1 + JW_1, A(V_2) + C_2(W_2) + J(B(W_2) + C_1(V_2)) \rangle_{1,1,0} \\ &= \langle V_1, A(V_2) + C_2(W_2) \rangle_{\mathfrak{k}} + \langle W_1, B(W_2) + C_1(V_2) \rangle_{\mathfrak{k}} \\ &= \langle V_1, A(V_2) \rangle_{\mathfrak{k}} + \langle W_1, B(W_2) \rangle_{\mathfrak{k}} + \langle V_1, C_2(W_2) \rangle_{\mathfrak{k}} + \langle W_1, C_1(V_2) \rangle_{\mathfrak{k}}. \end{aligned} \quad (3.4)$$

By assumption, $\langle \cdot, \cdot \rangle$ is $\mathrm{Ad}(K)$ -invariant. Coupled with the fact that Ad_k commutes with J for all $k \in K$ (cf. Lemma 2.5), we therefore have

$$\begin{aligned} & \langle V_1, A(V_2) \rangle_{\mathfrak{k}} + \langle W_1, B(W_2) \rangle_{\mathfrak{k}} + \langle V_1, C_2(W_2) \rangle_{\mathfrak{k}} + \langle W_1, C_1(V_2) \rangle_{\mathfrak{k}} \\ &= \langle V_1 + JW_1, V_2 + JW_2 \rangle = \langle \mathrm{Ad}_k V_1 + J\mathrm{Ad}_k W_1, \mathrm{Ad}_k V_2 + J\mathrm{Ad}_k W_2 \rangle \\ &= \langle \mathrm{Ad}_k V_1, A(\mathrm{Ad}_k V_2) \rangle_{\mathfrak{k}} + \langle \mathrm{Ad}_k W_1, B(\mathrm{Ad}_k W_2) \rangle_{\mathfrak{k}} \\ & \quad + \langle \mathrm{Ad}_k V_1, C_2(\mathrm{Ad}_k W_2) \rangle_{\mathfrak{k}} + \langle \mathrm{Ad}_k W_1, C_1(\mathrm{Ad}_k V_2) \rangle_{\mathfrak{k}} \end{aligned}$$

for all $k \in K$.

Consider the case $(W_1, W_2) = (0, 0)$. This yields

$$\langle V_1, A(V_2) \rangle_{\mathfrak{k}} = \langle \mathrm{Ad}_k V_1, A(\mathrm{Ad}_k V_2) \rangle_{\mathfrak{k}}, \quad \forall k \in K. \quad (3.5)$$

At the same time, $\langle \cdot, \cdot \rangle_{\mathfrak{k}}$ is $\mathrm{Ad}(K)$ -invariant, and so we also have

$$\langle V_1, A(V_2) \rangle_{\mathfrak{k}} = \langle \mathrm{Ad}_k V_1, \mathrm{Ad}_k A(V_2) \rangle_{\mathfrak{k}}, \quad \forall k \in K. \quad (3.6)$$

Combining (3.5) and (3.6), applied with the vector $V_1 = \text{Ad}_{k^{-1}}X$ for any $X \in \mathfrak{k}$ (and relabeling $V_2 = V$) yields

$$\langle X, A(\text{Ad}_k V) \rangle_{\mathfrak{k}} = \langle X, \text{Ad}_k A(V) \rangle_{\mathfrak{k}}, \quad \forall k \in K \text{ and } X, V \in \mathfrak{k}.$$

Since $\langle \cdot, \cdot \rangle_{\mathfrak{k}}$ is an inner product on \mathfrak{k} , this implies that $\text{Ad}_k A = A \text{Ad}_k$ for all $k \in K$; i.e. the endomorphism $A \in \text{End}(\mathfrak{k})$ commutes with the Adjoint representation $(\text{Ad}, \mathfrak{k})$ of \mathfrak{k} .

The stated assumption is that K is simple, which by definition (cf. [15, Definition 3.11]) means that \mathfrak{k} has no nontrivial ideals. It follows that the Adjoint representation $(\text{Ad}, \mathfrak{k})$ is irreducible. Indeed, if $\mathcal{J} \subset \mathfrak{k}$ is an invariant subspace for Ad , then $\text{Ad}_{e^{tX}}(Y) \in \mathcal{J}$ for all $X \in \mathfrak{k}$ and $Y \in \mathcal{J}$. Taking the derivative at $t = 0$ shows that $\text{ad}_X(Y) = [X, Y] \in \mathcal{J}$ for all $X \in \mathfrak{k}$ and $Y \in \mathcal{J}$, which means $\mathcal{J} \subset \mathfrak{k}$ is an ideal. Thus $\mathcal{J} \in \{0, \mathfrak{k}\}$, showing $(\text{Ad}, \mathfrak{k})$ is irreducible. In particular, since A commutes with this representation, by Schur's lemma, there is a complex number a so that $A = aI$.

Considering the cases $(V_1, V_2) = (0, 0)$, $(V_2, W_1) = (0, 0)$, and $(V_1, W_2) = (0, 0)$ in turn, precisely the same argument shows that there are constants $b, c_1, c_2 \in \mathbb{C}$ with $B = bI$, $C_1 = c_1 I$, and $C_2 = c_2 I$. Hence, from (3.4), we have

$$\langle V_1 + JW_1, V_2 + JW_2 \rangle = a\langle V_1, V_2 \rangle_{\mathfrak{k}} + b\langle W_1, W_2 \rangle_{\mathfrak{k}} + c_1\langle W_1, V_2 \rangle_{\mathfrak{k}} + c_2\langle V_1, W_2 \rangle_{\mathfrak{k}}. \quad (3.7)$$

Since $\langle \cdot, \cdot \rangle$ is symmetric, if we interchange $V_1 \leftrightarrow V_2$ and $W_1 \leftrightarrow W_2$, the value of (3.7) is unchanged. Using the symmetry of $\langle \cdot, \cdot \rangle_{\mathfrak{k}}$ it thus follows that

$$c_1\langle W_1, V_2 \rangle_{\mathfrak{k}} + c_2\langle V_1, W_2 \rangle_{\mathfrak{k}} = c_2\langle W_1, V_2 \rangle_{\mathfrak{k}} + c_1\langle V_1, W_2 \rangle_{\mathfrak{k}}, \quad \forall V_1, V_2, W_1, W_2 \in \mathfrak{k}$$

and applying this in the case $(V_1, W_2) = 0$ while $W_1 = V_2$ is a $\langle \cdot, \cdot \rangle_{\mathfrak{k}}$ -unit vector shows that $c_1 = c_2$. Denoting this common value as $c_1 = c_2 = \frac{c}{2}$, (3.7) therefore shows that $\langle \cdot, \cdot \rangle$ has the form given in (3.1), i.e. $\langle \cdot, \cdot \rangle = \langle \cdot, \cdot \rangle_{a,b,c}$, where a, b, c are *a priori* complex numbers.

Taking $(W_1, W_2) = 0$ and $V_1 = V_2$ a $\langle \cdot, \cdot \rangle_{\mathfrak{k}}$ -unit vector shows that $a > 0$; taking $(V_1, V_2) = (0, 0)$ and $W_1 = W_2$ a $\langle \cdot, \cdot \rangle_{\mathfrak{k}}$ -unit vector shows that $b > 0$; and taking $V_1 = V_2 = tV$ and $W_1 = W_2 = V$ for any $\langle \cdot, \cdot \rangle_{\mathfrak{k}}$ -unit vector $V \in \mathfrak{k}$ and any $t \in \mathbb{R}$ yields

$$0 < \langle tV + JV, tV + JV \rangle = at^2 + b + 2ct, \quad \forall t \in \mathbb{R}.$$

Therefore, c must be a real number, and this quadratic polynomial must have negative discriminant: $(2c)^2 - 4ab < 0$, and it follows that $c^2 < ab$.

Finally, if $\langle \cdot, \cdot \rangle'_{\mathfrak{k}}$ is another Ad -invariant inner product on \mathfrak{k} , then it can be extended to an $\text{Ad}(K)$ -invariant inner product on $\mathfrak{k}_{\mathbb{C}}$ (for example as in (3.3)); this extension then must have the form (3.1) as proven above. Restricting back to \mathfrak{k} shows that $\langle \cdot, \cdot \rangle'_{\mathfrak{k}} = a\langle \cdot, \cdot \rangle_{\mathfrak{k}}$ for some $a > 0$. Thus the original Ad -invariant inner product on \mathfrak{k} is unique up to scale, and this concludes the proof. \square

Remark 3.3. The question posed at the beginning of this section, regarding the set of $\text{Ad}(K)$ -invariant inner products on $\mathfrak{k}_{\mathbb{C}}$ that *extend* the given Ad -invariant inner product on \mathfrak{k} , is therefore answered by the two-parameter family (3.1) with $a = 1$.

We now turn to the Laplacian on $\mathfrak{k}_{\mathbb{C}}$ associated to the inner product (3.1). The notation $\Delta_{\mathfrak{k}_{\mathbb{C}}}$ for this Laplacian (cf. Section 2.5) is lacking, as it depends upon the specified inner product on $\mathfrak{k}_{\mathbb{C}}$ without notation to refer to it. Thus, we refer to the Laplacian in this case as $L_{a,b,c}$. Lemma 2.23 allows us to compute it quickly.

Proposition 3.4. *Let $L_{a,b,c}$ denote the Laplacian on $C^2(K_{\mathbb{C}})$ associated to the inner product $\langle \cdot, \cdot \rangle_{a,b,c}$ of (3.1). Fix any basis $\{X_j\}_{j=1}^d$ of \mathfrak{k} orthonormal with respect to the given*

$\text{Ad}(K)$ -invariant inner product on \mathfrak{k} , and let $Y_j = JX_j$. Then

$$L_{a,b,c} = \frac{1}{ab-c^2} \sum_{j=1}^d \left[b\tilde{X}_j^2 + a\tilde{Y}_j^2 - 2c\tilde{X}_j\tilde{Y}_j \right]. \quad (3.8)$$

Proof. Since $\mathfrak{k}_{\mathbb{C}} = \mathfrak{k} \oplus J\mathfrak{k}$ (cf. Proposition 2.6), the set $\{X_j, Y_j\}_{j=1}^d$ is a basis for $\mathfrak{k}_{\mathbb{C}}$. Let $V_{2k-1} = X_k$ and $V_{2k} = Y_k$ for $1 \leq k \leq d$, and define $q_{ij} = \langle V_i, V_j \rangle_{a,b,c}$. By Lemma 2.23,

$$L_{a,b,c} = \sum_{i,j=1}^d q_{ij}^{-1} \tilde{V}_i \tilde{V}_j. \quad (3.9)$$

We can compute directly from (3.1) and the orthonormality of $\{X_j\}_{j=1}^d$ that

$$\langle X_i, X_j \rangle_{a,b,c} = a\delta_{ij}, \quad \langle Y_i, Y_j \rangle_{a,b,c} = b\delta_{ij}, \quad \langle X_i, Y_j \rangle_{a,b,c} = \langle Y_i, X_j \rangle_{a,b,c} = c\delta_{ij}.$$

It follows that the matrix q is block diagonal with 2×2 diagonal blocks all equal to the matrix B (below). Thus q^{-1} is also block diagonal with 2×2 diagonal blocks all equal to B^{-1} (below).

$$B = \begin{bmatrix} a & c \\ c & b \end{bmatrix}, \quad B^{-1} = \frac{1}{ab-c^2} \begin{bmatrix} b & -c \\ -c & a \end{bmatrix}.$$

Combining this with (3.9) yields (3.8). \square

To dispense with the cumbersome determinant in the denominator in (3.8), and match the parametrization relevant to the Segal-Bargmann transform, we make the following change of parametrization:

$$(s, t, u) = \Phi(a, b, c) := \frac{1}{ab-c^2}(a+b, 2a, 2c). \quad (3.10)$$

It is straightforward to verify that Φ is a diffeomorphism

$$\Phi: \{(a, b, c): a, b > 0, c^2 < ab\} \rightarrow \{(s, t, u): t > 0, u \in \mathbb{R}, 2s > t + u^2/t\}$$

with inverse

$$(a, b, c) = \Phi^{-1}(s, t, u) = \frac{4}{2st - t^2 - u^2} \left(\frac{t}{2}, s - \frac{t}{2}, \frac{u}{2} \right) = \frac{1}{\alpha} \left(\frac{t}{2}, s - \frac{t}{2}, \frac{u}{2} \right)$$

referring to the constant α if (1.7), which is > 0 precisely in range of Φ . From here on, we use the parameters (s, t, u) . This prompts us to introduce the following notation for a reparametrized and rescaled Laplace operator.

Definition 3.5. Let $\tau = t + iu \in \mathbb{C}$ and $s > 0$ satisfy (1.6). The (s, τ) -**Laplacian** $\Delta_{s,\tau}$ on $K_{\mathbb{C}}$ is the elliptic operator

$$\Delta_{s,\tau} = \sum_{j=1}^d \left[\left(s - \frac{t}{2} \right) \tilde{X}_j^2 + \frac{t}{2} \tilde{Y}_j^2 - u \tilde{X}_j \tilde{Y}_j \right]$$

where $\{X_j\}_{j=1}^d$ is any orthonormal basis of \mathfrak{k} , and $Y_j = JX_j$.

Note: $\Delta_{s,\tau} = \alpha \cdot L_{a,b,c}$ for $(a, b, c) = \Phi^{-1}(s, t, u)$. Since $L_{a,b,c}$ is elliptic for this parameter range (cf. Theorem 2.30 and Proposition 3.4) and $\alpha > 0$, it follows that $\Delta_{s,\tau}$ is elliptic in the indicated parameter range.

3.2. Invariant Heat Kernels on K and $K_{\mathbb{C}}$. Let us now fix notation for the heat kernels relevant to the Segal–Bargmann transform.

Definition 3.6. Let K be a compact type Lie group, with a fixed $\text{Ad}(K)$ -invariant inner product. For $t > 0$, denote by $\Delta_{\mathfrak{k}}$ the associated Laplacian, and by $\rho_t: K \rightarrow \mathbb{R}_+$ the associated heat kernel (cf. Theorem 2.30). In addition, for $\tau \in \mathbb{C}_+$ and $s > 0$ satisfying (1.6), denote by $\mu_{s,\tau}: K_{\mathbb{C}} \rightarrow \mathbb{R}_+$ the heat kernel (at time 1) associated to the Laplacian $\Delta_{s,\tau}$ of Definition 3.5.

To be clear: following Proposition 2.33, ρ_t is the integral kernel of the heat operator $e^{\frac{t}{2}\Delta_{\mathfrak{k}}}$, acting on either $L^2(K)$ or $L^2(K_{\mathbb{C}})$:

$$e^{\frac{t}{2}\Delta_{\mathfrak{k}}} = \int_K \rho_t(k) \hat{R}_k dk.$$

Similarly, $\mu_{s,\tau}$ is the integral kernel of the heat operator for the Laplacian $\Delta_{s,\tau}$:

$$e^{\frac{1}{2}\bar{\Delta}_{s,\tau}} = \int_{K_{\mathbb{C}}} \mu_{s,\tau}(z) \hat{R}_z dz.$$

(We could include an additional time parameter $(e^{\frac{r}{2}\bar{\Delta}_{s,\tau}})_{r>0}$ here, but following Definition 3.5 we see that $r\Delta_{s,\tau} = \Delta_{rs,r\tau}$, so there is no loss in absorbing this extra parameter into ones already present.)

Note: with $\tau = t + iu$, $\Delta_{s,\tau}$ is the Laplacian on $K_{\mathbb{C}}$ determined by the inner product $\alpha\langle \cdot, \cdot \rangle_{a,b,c}$ where $\alpha > 0$ (cf. (1.7)) and $(a, b, c) = \frac{1}{\alpha}(\frac{t}{2}, s - \frac{t}{2}, \frac{u}{2})$. In particular, it is the Laplacian for an $\text{Ad}(K)$ -invariant inner product. Therefore Corollary 2.32 applies:

$$\begin{aligned} [e^{\frac{1}{2}\bar{\Delta}_{s,\tau}}, \hat{R}_k] &= 0 \\ \mu_{s,\tau}(kzk^{-1}) &= \mu_{s,\tau}(z) \end{aligned} \quad \text{for } z \in K_{\mathbb{C}}, k \in K. \quad (3.11)$$

Also Theorem 2.34 applies: $e^{\frac{r}{2}\bar{\Delta}_{\mathfrak{k}}}$ and $e^{\frac{1}{2}\bar{\Delta}_{s,\tau}}$ commute for all r, s, τ .

For the remainder of this section, we assume K is compact. In that case, Theorem 2.48 applies to $\mu_{s,\tau}$: there is a constant $C = C(s, \tau)$ such that

$$C^{-1}\mu_{s,\tau}(zk) \leq \mu_{s,\tau}(z) \leq C\mu_{s,\tau}(zk), \quad \forall k \in K, z \in K_{\mathbb{C}}. \quad (3.12)$$

Averaging over $k \in K$, as in Corollary 2.49, we see that $\mu_{s,\tau}$ is equivalent to its K -average. We will do this averaging using the Haar measure, and therefore introduce the following K -averaged version of $\mu_{s,\tau}$.

Definition 3.7. Let $\tau = t + iu \in \mathbb{C}_+$ and $s > 0$ satisfy (1.6). Define the K -averaged heat kernel ν_t on $K_{\mathbb{C}}$ as

$$\nu_t(z) = \int_K \mu_{s,\tau}(zk) dk.$$

The notation for ν_t , without explicit reference to the other two parameters s and u , is justified by the following lemma.

Lemma 3.8. The function $\nu_t: K_{\mathbb{C}} \rightarrow \mathbb{R}_+$ is independent of s and depends on τ only through $t = \text{Re } \tau$.

Proof. Let $f \in C_c^\infty(K_\mathbb{C})$. Then we can apply Fubini's theorem twice and integrate as follows.

$$\begin{aligned}
\int f(z) \nu_t(z) dz &= \int_G f(z) \left[\int_K \mu_{s,\tau}(zk) dk \right] dz \\
&= \int_K \left[\int_G \mu_{s,\tau}(zk) f(z) dz \right] dk \\
&= \int_K \left[\int_G \mu_{s,\tau}(w) f(wk^{-1}) dy \right] dk \\
&= \int_G \mu_{s,\tau}(w) \left[\int_K f(wk^{-1}) dk \right] dw = \int_G \hat{f}(w) \mu_{s,\tau}(w) dw \quad (3.13)
\end{aligned}$$

where $\hat{f}(w) = \int_K f(wk^{-1}) dk = \int_K f(wk) dk$ is the K -average of f . Since K is compact, \hat{f} is still compactly supported, and by Theorem 2.30,

$$\int_G \hat{f}(w) \mu_{s,\tau}(w) dw = \left(e^{\frac{1}{2}\Delta_{s,\tau}} \hat{f} \right) (e).$$

Now, the function \hat{f} is, by definition, K -right invariant: $\hat{R}_k \hat{f} = \hat{f}$ for $k \in K$. In particular, if $X \in \mathfrak{k}$, $\hat{f}(ze^{rX}) = \hat{f}(z)$ for all $z \in K_\mathbb{C}$ and $r \in \mathbb{R}$. Differentiating at $r = 0$ shows that $\tilde{X}\hat{f} = 0$ for any $X \in \mathfrak{k}$. Appealing to Definition 3.5 and writing $\tau = t + iu$, we then have

$$\Delta_{s,\tau} \hat{f} = \sum_{j=1}^d \left[\left(s - \frac{t}{2} \right) \tilde{X}_j^2 + \frac{t}{2} \tilde{Y}_j^2 - u \tilde{X}_j \tilde{Y}_j \right] \hat{f} = \frac{t}{2} \sum_{j=1}^d \tilde{Y}_j^2 \hat{f}$$

where we have used the fact that $\tilde{X}_j \tilde{Y}_j = \tilde{Y}_j \tilde{X}_j$, which follows from the definition $Y_j = JX_j$ and Corollary 2.3. By Theorem 2.26, the operator $\Delta_{i\mathfrak{k}} := \sum_{j=1}^d \tilde{Y}_j^2$ is essentially self-adjoint with core $C_c^\infty(K_\mathbb{C})$. In total, then, we have shown that

$$\int f(z) \nu_t(z) dz = \left(e^{\frac{t}{4}\Delta_{i\mathfrak{k}}} \hat{f} \right) (e), \quad \text{for } f \in L^2(K_\mathbb{C}). \quad (3.14)$$

This is explicitly independent of s and u for any f . By letting f run through an approximate identity sequence centered at any point $z \in K_\mathbb{C}$, the limit shows that $\nu_t(z)$ is independent of s and u as claimed. \square

The K -averaged heat kernel will appear in an essential way in Section 4.2.4. It will also arise as a technical tool in Section 3.3 below, in the context of Corollary 2.49, which says that $\mu_{s,\tau}$ is equivalent to ν_t (with constants depending on s, τ): taking $C = C(s, \tau)$ as in (3.12),

$$C^{-1} \nu_t(z) \leq \mu_{s,\tau}(z) \leq C \nu_t(z), \quad \text{for all } z \in K_\mathbb{C}. \quad (3.15)$$

3.3. A Density Theorem for $\mathcal{H}L^2(K_\mathbb{C}, \mu_{s,\tau})$. Let G be a complex Lie group. For a given measure η on G , consider the space $\mathcal{H}L^2(G, \eta)$ of holomorphic functions that are in $L^2(G, \eta)$. In general, it is not a priori clear if this space contains any non-constant functions. Presently, our goal is to show that, with the measure $\eta = \mu_{s,\tau}(x) dx$, the holomorphic L^2 space (which we denote $\mathcal{H}L^2(K_\mathbb{C}, \mu_{s,\tau})$) is a rich space.

In the case $K = \mathbb{R}^k$ and $(s, \tau) = (1, 1)$, $\mu_{1,1} = \gamma$ is a centered Gaussian measure on \mathbb{C}^k . It is elementary to verify that all holomorphic polynomials are in $\mathcal{H}L^2(\mathbb{C}^k, \gamma)$; moreover, it is a theorem that they are dense. It is instructive to outline why this is true (for simplicity in the case $k = 1$). If $f \in \mathcal{H}(\mathbb{C})$ is an entire holomorphic function, the Taylor polynomials

of f centered at 0 converge to f uniformly on compact sets, and hence pointwise. At the same time, the rotational invariance of the Gaussian measure γ shows that monomials $\{z^n\}_{n \in \mathbb{N}}$ are orthogonal in $L^2(\mathbb{C}, \gamma)$. This, together with growth estimates, easily shows that the Taylor polynomials also converge to f in L^2 . The pointwise limit must agree with the L^2 limit, showing that f can be approximated in L^2 by polynomials.

Following [16], we now prove that the analog of this approach holds in general for $K_{\mathbb{C}}$ and $\mu_{s,\tau}$, in the case that K is compact, and polynomials are replaced by matrix entries (cf. Definition 2.36). Note, by (3.15), as vector spaces $L^2(K_{\mathbb{C}}, \mu_{s,\tau}) = L^2(K_{\mathbb{C}}, \nu_t)$ (with equivalent but not equal norms); thus we may just as well work with the K -averaged measure ν_t when K is compact.

For compact K , the Peter–Weyl theorem asserts that the matrix entries on K are dense in $L^2(K, dx)$; since the heat kernel ρ_t on K is bounded and bounded above 0 for $t > 0$ (as in the proof of Theorem 2.48), $L^2(K, dx)$ and $L^2(K, \rho_t)$ are equal as vector spaces, and so matrix entries are also dense in $L^2(K, \rho_t)$. The same may not be true for $L^2(K_{\mathbb{C}}, \mu_{s,\tau})$; but, as in the Gaussian case above, we will show that the *holomorphic* matrix entries are dense in $\mathcal{H}L^2(K_{\mathbb{C}}, \mu_{s,\tau})$. The proof is more or less precisely the same as the one given in [16, Section 8] (there for the special case $s = t$ and $u = 0$), so we presently only outline the steps.

If K is compact, denote by \hat{K} the set of (equivalence classes of) irreducible representations of K (which are necessarily finite dimensional in the compact setting). For $f \in L^2(K, dx)$, the Peter–Weyl theorem gives a “Fourier expansion” for f : as an L^2 sum,

$$f(x) = \sum_{\pi \in \hat{K}} \text{Tr}(\pi(x) A_{\pi,f}) \quad (3.16)$$

where the Fourier coefficients $A_{\pi,f}$ are endomorphisms of V_{π} determined by the formula

$$A_{\pi,f} = \dim V_{\pi} \int_K f(x) \pi(x^{-1}) dx. \quad (3.17)$$

Moreover, if f is smooth on K , then the series (3.16) converges uniformly on K ; this is proved using explicit bounds on the growth of matrix entries, in [16, Sections 4 & 8].

For any $\pi \in \hat{K}$, we may view the range space $\pi(K) \subseteq \text{GL}(V_{\pi})$ as living in the complex Lie group $\text{GL}(V_{\pi} \otimes_{\mathbb{R}} \mathbb{C})$. From the universal property defining the complexification $K_{\mathbb{C}}$ (cf. Section 2.1), there is a holomorphic homomorphism $\pi_{\mathbb{C}}: K_{\mathbb{C}} \rightarrow \text{GL}(V_{\pi} \otimes_{\mathbb{R}} \mathbb{C})$ such that $\pi_{\mathbb{C}}(x) = \pi(x)$ for all $x \in K$. Thus, each of the terms $x \mapsto \text{Tr}(\pi(x) A_{\pi,f})$ in (3.16) has an analytic continuation $z \mapsto \text{Tr}(\pi_{\mathbb{C}}(z) A_{\pi,f})$ to a holomorphic matrix entry function on $K_{\mathbb{C}}$.

Now, let $F \in \mathcal{H}(K_{\mathbb{C}})$ (not necessarily in L^2). Then $F|_K$ is smooth, hence in $L^2(K, dx)$, and so it has a Fourier expansion (3.16). Analytically continuing each of the terms in the series, we putatively introduce the *holomorphic Fourier expansion*

$$K_{\mathbb{C}} \ni z \mapsto \sum_{\pi \in \hat{K}} \text{Tr}(\pi_{\mathbb{C}}(z) A_{\pi,F}). \quad (3.18)$$

At the moment this is a formal series; for $z = k \in K$ it converges (in L^2 sense) to $F(k)$, but for $z \in K_{\mathbb{C}}$ in general it is not clear a priori that it converges in any sense. In fact:

Lemma 3.9 ([16], Lemma 9). *For any holomorphic function $F \in \mathcal{H}(K_{\mathbb{C}})$, the holomorphic Fourier series (3.18) converges to F uniformly on compact subset of $K_{\mathbb{C}}$.*

Proof. The idea of the proof of Lemma 3.9 is to compute the Fourier coefficients of the smooth function $F_g(z) = F(zg)$ for each $g \in K_{\mathbb{C}}$. By a change of variables in (3.17)

(first for $g \in K$ and then extended to $g \in K_{\mathbb{C}}$ by uniqueness of analytic continuation), it is straightforward to check that $A_{\pi, F_g} = \pi(g)A_{\pi, F}$. This shows that the putative holomorphic Fourier series (3.18) for F at the point zg is the same as the ordinary Fourier series (3.16) for F_g at the point x , which is known to converge pointwise since F_g is smooth. Finally, the same bounds used to prove uniform convergence of (3.16) on K show that (3.18) converges uniformly on compact subsets of $K_{\mathbb{C}}$. \square

To continue following the idea of the proof of the density of polynomials in $\mathcal{H}L^2(\mathbb{C}, \gamma)$, the next step is to show that the terms in (3.18) are orthogonal in $L^2(K_{\mathbb{C}}, \mu_{s, \tau})$. However, this need not be true: this requires an analog of the rotational invariance of γ . In this case, that means K -invariance, and so what we *can* prove is that (3.18) is an orthogonal sum that converges to F in $L^2(K_{\mathbb{C}}, \nu_t)$.

Lemma 3.10 ([16], Lemma 10). *For any holomorphic function $F \in \mathcal{H}L^2(K_{\mathbb{C}}, \nu_t)$, the holomorphic Fourier series converge to F in $L^2(K_{\mathbb{C}}, \nu_t)$.*

Proof. We first show that the terms in the holomorphic Fourier series (3.18) are orthogonal in $L^2(K_{\mathbb{C}}, \nu_t)$. If $\pi, \sigma \in \hat{K}$, $A \in \text{End}(V_{\pi})$ and $B \in \text{End}(V_{\sigma})$, we can use the K -invariance of ν_t to compute

$$\begin{aligned} \langle f_{\pi, A}, f_{\sigma, B} \rangle_{L^2(K_{\mathbb{C}}, \nu_t)} &= \int_{K_{\mathbb{C}}} \overline{\text{Tr}(\pi_{\mathbb{C}}(g)A)} \text{Tr}(\sigma_{\mathbb{C}}(g)B) \nu_t(g) dg \\ &= \int_K \int_{K_{\mathbb{C}}} \overline{\text{Tr}(\pi_{\mathbb{C}}(xg)A)} \text{Tr}(\sigma_{\mathbb{C}}(xg)B) \nu_t(g) dg dx \\ &= \int_{K_{\mathbb{C}}} \left[\int_K \overline{\text{Tr}(\pi(x)\pi_{\mathbb{C}}(g)A)} \text{Tr}(\sigma(x)\sigma_{\mathbb{C}}(g)B) dx \right] \nu_t(g) dg \\ &= \int_{K_{\mathbb{C}}} \langle f_{\pi, A(g)}, f_{\sigma, B(g)} \rangle_{L^2(K, dx)} \nu_t(g) dg \end{aligned} \quad (3.19)$$

where $A(g) = \pi_{\mathbb{C}}(g)A$ and $B(g) = \sigma_{\mathbb{C}}(g)B$. The second equality is justified by (3.13), which shows that $\int_{K_{\mathbb{C}}} f(g) \nu_t(g) dg = \int_{K_{\mathbb{C}}} \hat{f}(g) \nu_t(g) dg$ for any $f \in C_c(G)$ (where \hat{f} is the right- K -average of f), and extends easily to matrix entries f using Theorem 2.30(7), Lemma 2.40 and the dominated convergence theorem. The third equality's application of Fubini's theorem is also justified by Theorem 2.30(7) and Lemma 2.40. If π and σ are inequivalent irreducible representations, the Haar inner product in the integrand of (3.19) is 0 by the Peter–Weyl theorem. This yields the desired orthogonality.

Since the sum (3.18) is orthogonal, to prove convergence in $L^2(K_{\mathbb{C}}, \nu_t)$ it suffices to show that

$$\sum_{\pi \in \hat{K}} \|\text{Tr}(\pi_{\mathbb{C}}(\cdot)A_{\pi, F})\|_{L^2(K_{\mathbb{C}}, \nu_t)}^2 < \infty. \quad (3.20)$$

To prove this, fix a sequence $(E_n)_{n \in \mathbb{N}}$ of increasing compact left- K -invariant subsets with $\bigcup_n E_n = K_{\mathbb{C}}$. (To construct such a sequence, first select any exhaustion of the separable topological space $K_{\mathbb{C}}$ by compact sets \mathcal{E}_n . Then let $E_n = K \cdot \mathcal{E}_n$, the left-orbit of \mathcal{E}_n under K . The sets E_n are manifestly left- K -invariant, and increase to $K_{\mathbb{C}}$. E_n is compact since it is the image of the compact set $K \times \mathcal{E}_n$ under the continuous map $(x, g) \mapsto xg$.) Replacing $K_{\mathbb{C}}$ with E_n in the equations culminating in (3.19) does not affect the computation at all, due to the left- K -invariance. Hence the terms in (3.18) are orthogonal in $L^2(E_n, \nu_t|_{E_n})$.

By Lemma 3.9, (3.18) converges to F uniformly on compact subsets. Since $\nu_t|_{E_n}$ is a finite measure, it follows that (3.18) converges to $F|_{E_n}$ in $L^2(E_n, \nu_t)$. In particular, this

means that

$$\|F|_{E_n}\|_{L^2(E_n, \nu_t)}^2 = \sum_{\pi \in \hat{K}} \|\mathrm{Tr}(\pi_{\mathbb{C}}(\cdot) A_{\pi, F})|_{E_n}\|_{L^2(E_n, \nu_t)}^2. \quad (3.21)$$

Both sides of (3.21) are monotone increasing in n , and hence the monotone convergence theorem implies that we can take the limit, yielding

$$\sum_{\pi \in \hat{K}} \|\mathrm{Tr}(\pi_{\mathbb{C}}(\cdot) A_{\pi, F})\|_{L^2(K_{\mathbb{C}}, \nu_t)}^2 = \|F\|_{L^2(K_{\mathbb{C}}, \nu_t)}^2 < \infty$$

justifying (3.20).

Thus, the holomorphic Fourier series (3.18) converges in $L^2(K_{\mathbb{C}}, \nu_t)$. By Lemma 3.9, it also converges pointwise to F ; the pointwise limit must coincide with the L^2 limit, and so the holomorphic Fourier series converges to F in L^2 , as desired. \square

This brings us to the main result of this section.

Theorem 3.11. *Let $\tau \in \mathbb{C}_+$ and $s > 0$ satisfy (1.6). The holomorphic matrix entries on $K_{\mathbb{C}}$ are dense in $\mathcal{H}L^2(K_{\mathbb{C}}, \mu_{s, \tau})$.*

To be clear: *holomorphic matrix entries* are those matrix entries $f_{\pi, A}$ on $K_{\mathbb{C}}$ (cf. Definition 2.36) that are holomorphic functions $K_{\mathbb{C}} \rightarrow \mathbb{C}$. If (π, V_{π}) is a complex representation for which $\pi: K_{\mathbb{C}} \rightarrow \mathrm{GL}(V_{\pi})$ is a holomorphic map, then $f_{\pi, A}$ is holomorphic for any $A \in \mathrm{End}(V_{\pi})$.

Proof. By (3.15), the Hilbert spaces $L^2(K_{\mathbb{C}}, \mu_{s, \tau})$ and $L^2(K_{\mathbb{C}}, \nu_t)$ are equal as sets, with equivalent norms; in particular, membership and convergence in the two spaces are the same. Thus, for $F \in L^2(K_{\mathbb{C}}, \mu_{s, \tau})$, by Lemma 3.10, the holomorphic Fourier series (3.18) converges to F in $L^2(K_{\mathbb{C}}, \nu_t)$, and hence also in $L^2(K_{\mathbb{C}}, \mu_t)$. The terms in this series are all holomorphic matrix entries; this concludes the proof. \square

4. THE SEGAL–BARGMANN TRANSFORM

In this section, we prove that the heat kernel $\rho_t(x)$ on a compact type Lie group has an analytic continuation in both the space variable x and the time variable t , and that the integral operator it defines, the Segal–Bargmann transform (1.5), is an isometric isomorphism. Utilizing Proposition 2.10 to decompose the group as a product of a compact group and a copy of \mathbb{R}^k , we will largely restrict our calculations to these two separate cases, with comments about combining them in Section 4.3.

4.1. The Euclidean Case. Throughout this section, we work on a Euclidean space \mathbb{R}^k for some $k \in \mathbb{N}$. In this setting, there is an explicit expression for both the analytically continued heat kernel $\rho_{\mathbb{C}}$, and the three-parameter heat kernel density $\mu_{s, \tau}$, which we develop below in Sections 4.1.1 and 4.1.3. In Section 4.1.2, we show that the integral kernel definition (1.5) of the transform $B_{s, \tau} f$ converges for $f \in L^2(\mathbb{R}^k, \rho_s)$, when τ and s satisfy (1.6). This argument is not quite fine enough to prove that the resulting function is in $L^2(\mathbb{C}^k, \mu_{s, \tau})$; instead, in Section 4.1.4 we prove that the transform is an isometry when restricted to polynomials, which are dense in $L^2(\mathbb{R}^k, \rho_s)$. Section 4.1.5 is then devoted to the technical details of extending this isometry to the full domain (without prior knowledge that it is bounded), along with the proof that it is surjective onto the holomorphic Hilbert space $\mathcal{H}L^2(\mathbb{C}^k, \mu_{s, \tau})$. Finally, Section 4.1.6 outlines an alternate proof, reducing to the previously known two-parameter case [8, 16] via a change of variables.

4.1.1. *Analytic Continuation of the Heat Kernel.* In the classical setting, the heat kernel ρ_s on \mathbb{R}^k is explicitly known to be the Gaussian density mentioned in the introduction:

$$\rho_s(x) = (2\pi s)^{-k/2} \exp\left(-\frac{|x|^2}{2s}\right).$$

Here, the analytic continuation in both variables is immediate. For $\tau \in \mathbb{C}_+$ expressed in polar form $\tau = re^{i\theta}$ with $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$, define $\sqrt{\tau} = \sqrt{r}e^{i\theta/2}$. (This is the standard branch of the square root, which is holomorphic on $\mathbb{C} \setminus \mathbb{R}_- \supset \mathbb{C}_+$.) Then, for $\tau \in \mathbb{C}_+$ and $z \in \mathbb{C}^k$, we define

$$\rho_{\mathbb{C}}(\tau, z) := \left(\sqrt{2\pi\tau}\right)^{-k} \exp\left(-\frac{z \cdot z}{2\tau}\right) \quad (4.1)$$

where $z \cdot z = \sum_{j=1}^k z_j^2$. The function $\rho_{\mathbb{C}}$ is evidently holomorphic on $\mathbb{C}_+ \times \mathbb{C}^k$, restricts to $\rho_t(x)$ when $\tau = t \in \mathbb{R}_+$ and $z = x \in \mathbb{R}^k$, and is the unique such analytic continuation (since the set $\mathbb{R}_+ \times \mathbb{R}^k$ accumulates in $\mathbb{C}_+ \times \mathbb{C}^k$). This concludes the proof of Theorem 1.1 in this case.

4.1.2. *The Transform $B_{s,\tau}$ is Well-Defined.* Now, consider the putative definition of the transform $B_{s,\tau}f$ in (1.5).

Proposition 4.1. *Let $\tau = t + iu \in \mathbb{C}_+$ and $s > 0$ satisfy (1.6). If $f \in L^2(\mathbb{R}^k, \rho_s)$, then the integral*

$$(B_{s,\tau}f)(z) = \int_{\mathbb{R}^k} f(v) \rho_{\mathbb{C}}(\tau, z - v) dv$$

converges for all $z \in \mathbb{C}^k$, and satisfies the pointwise bound

$$|(B_{s,\tau}f)(x + iy)| \leq \left(\frac{\pi s}{\sqrt{\alpha}}\right)^{k/2} \exp\left(\frac{t/2}{4\alpha}|x|^2 + \frac{s - t/2}{4\alpha}|y|^2 + \frac{u}{4\alpha}x \cdot y\right) \|f\|_{L^2(\mathbb{R}^k, \rho_s)}$$

where $\alpha > 0$ is the constant in (1.7).

Proof. We rewrite the definition of the transform as follows.

$$(B_{s,\tau}f)(z) = \int_{\mathbb{R}^k} \frac{\rho_{\mathbb{C}}(\tau, z - v)}{\rho_s(v)} f(v) \rho_s(v) dv.$$

Now applying the Cauchy–Schwarz inequality with respect to the measure $\rho_s(v) dv$ yields

$$|(B_{s,\tau}f)(z)| \leq \left(\int_{\mathbb{R}^k} \frac{|\rho_{\mathbb{C}}(\tau, z - v)|^2}{\rho_s(v)} dv\right)^{1/2} \|f\|_{L^2(\mathbb{R}^k, \rho_s)} \quad (4.2)$$

where the denominator in the integral appears in the form $\frac{1}{\rho_s(v)^2} \rho_s(v)$. Now,

$$\begin{aligned} |\rho_{\mathbb{C}}(\tau, z - v)|^2 &= (2\pi|\tau|)^{-k} |e^{-z \cdot z/\tau}| |e^{2v \cdot z/\tau}| |e^{-|v|^2/\tau}| \\ &= (2\pi|\tau|)^{-k} e^{-\operatorname{Re}(z \cdot z/\tau)} e^{2v \cdot \operatorname{Re}(z/\tau) - |v|^2 \operatorname{Re}(1/\tau)}. \end{aligned}$$

In terms of $z = x + iy$ and $\tau = t + iu$, we have

$$\operatorname{Re}(z/\tau) = \frac{tx + uy}{|\tau|^2}, \quad \operatorname{Re}(1/\tau) = \frac{t}{|\tau|^2}.$$

Hence, the integrand in (4.2), accounting for the normalization coefficients, is

$$\left(\frac{2\pi s}{|\tau|^2}\right)^{k/2} e^{-\operatorname{Re}(z \cdot z/\tau)} \exp\left\{\frac{1}{|\tau|^2}(2(tx + uy) \cdot v - t|v|^2) + \frac{1}{2s}|v|^2\right\}. \quad (4.3)$$

This Gaussian integral can be computed exactly. First note that the coefficient of $-|v|^2$ is

$$\frac{t}{|\tau|^2} - \frac{1}{2s} = \frac{2st - |\tau|^2}{2s|\tau|^2} = \frac{2\alpha}{s|\tau|^2} > 0,$$

cf. (1.7). Then we compute that

$$\int_{\mathbb{R}^k} \exp \left\{ -\frac{2\alpha}{s|\tau|^2} |v|^2 + \frac{2(tx + uy)}{|\tau|^2} \cdot v \right\} dv = \left(\frac{\pi s |\tau|^2}{2\alpha} \right)^{k/2} \exp \left\{ \frac{s}{2\alpha |\tau|^2} |tx + uy|^2 \right\}.$$

Combining this with (4.3), and using the identity $\operatorname{Re}(z \cdot z/\tau) = \frac{t(|x|^2 - |y|^2) + 2ux \cdot y}{|\tau|^2}$, we see that the integral in (4.2) is equal to

$$\frac{(\pi s)^k}{\alpha^{k/2}} \exp \left\{ \frac{1}{|\tau|^2} \left(-t|x|^2 + t|y|^2 - 2ux \cdot y + \frac{s}{2\alpha} |tx + uy|^2 \right) \right\}.$$

Elementary computations show that the expression in the exponential simplifies to

$$\frac{t/2}{2\alpha} |x|^2 + \frac{s - t/2}{2\alpha} |y|^2 + \frac{u}{2\alpha} x \cdot y$$

and the result now follows. \square

Remark 4.2. A more involved version of this argument, estimating not only the modulus of $\rho_{\mathbb{C}}(\tau, z)$ but also its partial derivatives in the space variables, allows us to justify (by the dominated convergence theorem) differentiating $\frac{\partial}{\partial \bar{z}}$ under the integral in (1.5) to show that the resulting function is holomorphic on \mathbb{C}^k . We will apply a simplified form of this argument to deduce the analyticity of $B_{s,\tau} f$ Section 4.1.4.

4.1.3. The Spaces $L^2(\mathbb{R}^k, \rho_s)$ and $\mathcal{H}L^2(\mathbb{C}^k, \mu_{s,\tau})$. The domain of the Segal–Bargmann transform $B_{s,\tau}$ is the space $L^2(\mathbb{R}^k, \rho_s)$ where ρ_s is the standard Gaussian measure of variance s . This Hilbert space is extremely well known. It has an orthonormal basis consisting of tensor products of Hermite polynomials of variance s ; in particular, polynomials are dense in $L^2(\mathbb{R}^k, \rho_s)$.

As we will shortly show, the range $B_{s,\tau}(L^2(\mathbb{R}^k, \rho_s))$ is the holomorphic Hilbert space $\mathcal{H}L^2(\mathbb{C}^k, \mu_{s,\tau})$. To study it, we begin with a complete description of the heat kernel $\mu_{s,\tau}$ in the Euclidean setting. Taking the standard orthonormal basis $\{X_j\}_{j=1}^k$ for \mathbb{R}^k , the associated vector fields are $\tilde{X}_j = \frac{\partial}{\partial x_j}$. The complexification is \mathbb{C}^k , and $\tilde{Y}_j = \widetilde{JX_j} = \frac{\partial}{\partial y_j}$. Thus, from Definition 3.5, with $\tau = t + iu$ we have

$$\Delta_{s,\tau} = \sum_{j=1}^k \left[\left(s - \frac{t}{2} \right) \frac{\partial^2}{\partial x_j^2} + \frac{t}{2} \frac{\partial^2}{\partial y_j^2} - u \frac{\partial^2}{\partial x_j \partial y_j} \right]. \quad (4.4)$$

Proposition 4.3. *Let $\tau = t + iu \in \mathbb{C}_+$ and $s > 0$ satisfy (1.6), and define $\alpha > 0$ as in (1.7). The complex heat kernel density $\mu_{s,\tau}$ on $(\mathbb{R}^k)_{\mathbb{C}} = \mathbb{C}^k$ of Definition 3.6 is given explicitly by*

$$\mu_{s,\tau}(x, y) = (2\pi\sqrt{\alpha})^{-k} \exp \left(-\frac{t/2}{2\alpha} |x|^2 - \frac{s - t/2}{2\alpha} |y|^2 - \frac{u}{2\alpha} x \cdot y \right). \quad (4.5)$$

Proof. By Theorem 2.30 (2), (4), and (5), $\mu_{s,\tau}(x, y) = \psi(1, x, y)$ where $\psi(r, x, y)$ is a probability density that solves the heat equation $\partial_r \psi = \frac{1}{2} \Delta_{s,\tau} \psi$, and tends to δ_0 as $\tau \downarrow 0$. On Euclidean space, these conditions uniquely specify the function $\psi(r, x, y)$. It is an elementary matter to verify that the function

$$\psi(r, x, y) = (2\pi r \sqrt{\alpha})^{-k} \exp \left(-\frac{t/2}{2\alpha r} |x|^2 - \frac{s - t/2}{2\alpha r} |y|^2 - \frac{u}{2\alpha r} x \cdot y \right)$$

satisfies these properties, and is a probability density. Letting $r = 1$ concludes the proof. \square

Remark 4.4. In the case $\tau = t > 0$, $\alpha = (s - \frac{t}{2})\frac{t}{2}$ and so (4.5) reduces to

$$\mu_{s,t}(x, y) = (\pi t)^{-k/2} (\pi(2s - t))^{-k/2} e^{-|x|^2/(2s-t)} e^{-|y|^2/t}.$$

This agrees with [8, Definition 3.1]. Under the additional assumption $s = t$, this reduces to the standard variance $t/2$ Gaussian density $(\pi t)^{-k} e^{-|z|^2/t}$ on \mathbb{C}^k .

Remark 4.5. Propositions 4.1 and 4.3 show that, for any $f \in L^2(\mathbb{R}^k, \rho_s)$, the transformed function $B_{s,\tau}f$ satisfies the pointwise bound

$$|B_{s,\tau}f(z)| \leq \left(\frac{s}{2\alpha}\right)^{k/2} \mu_{s,\tau}(z)^{-1/2} \|f\|_{L^2(\mathbb{R}^k, \rho_s)}.$$

This falls just short of showing that $B_{s,\tau}f$ is in $L^2(\mathbb{C}^k, \mu_{s,\tau})$. In fact, as we will soon show, $B_{s,\tau}$ is an isometry from $L^2(\mathbb{R}^k, \rho_s)$ onto $\mathcal{H}L^2(\mathbb{C}^k, \mu_{s,\tau})$, and the pointwise bound of Proposition 4.1 will play an important role in that proof.

The Gaussian density $\mu_{s,\tau}$ on \mathbb{C}^k is not rotationally invariant, and so (unlike the case $\mu_{t,t}$ which is the standard Gaussian density of variance $t/2$) monomial functions are typically not orthogonal in $\mathcal{H}L^2(\mathbb{C}^k, \mu_{s,\tau})$. Nevertheless, holomorphic polynomials are dense in this space. This is true for a generic nondegenerate Gaussian measure, as was proved in [8, Section 3.2]. We state the relevant facts here.

Proposition 4.6. *Let $\tau \in \mathbb{C}_+$ and $s > 0$ satisfy (1.6). The polynomial functions on \mathbb{R}^k are dense in $L^2(\mathbb{R}^k, \rho_s)$, and the holomorphic polynomial functions on \mathbb{C}^k are dense in $\mathcal{H}L^2(\mathbb{C}^k, \mu_{s,\tau})$.*

Proof. The density of polynomials in $L^2(\mathbb{R}^k, \rho_s)$ is well-known as mentioned above, and also follows from [8, Proposition 3.5]. The density of holomorphic polynomial functions in $\mathcal{H}L^2(\mathbb{C}^k, \mu_{s,\tau})$ is proved by an explicit Gram–Schmidt orthogonalization procedure that the authors call the *Hermite expansion*, and is a special case of [8, Theorem 3.6]. \square

Let us also highlight two important facts about the space $\mathcal{H}L^2(\mathbb{C}^k, \mu_{s,\tau})$.

Proposition 4.7. *The space $\mathcal{H}L^2(\mathbb{C}^k, \mu_{s,\tau})$ is a closed subspace of the Hilbert space $L^2(\mathbb{C}^k, \mu_{s,\tau})$. Moreover, for each $z \in \mathbb{C}^k$, the point evaluation linear functional given by $f \mapsto f(z)$ is continuous on $\mathcal{H}L^2(\mathbb{C}^k, \mu_{s,\tau})$.*

Proof. Both statements follow readily from the fact that this space possesses a reproducing kernel. Complete proofs (in a much more general context) can be found as [5, Theorem 3.2 & Corollary 3.3]. \square

4.1.4. Isometry on a Dense Subspace. In this section, we restrict our attention to polynomial functions. By Proposition 4.6, these form a dense subspace of the domain $L^2(\mathbb{R}^k, \rho_s)$. As discussed at the beginning of Section 2.7, and proved in greater generality in Proposition 2.45, the heat operator (interpreted as the convolution operator $f \mapsto f * \rho_s$) can be computed by the finitely-terminated power series expansion of the exponential $e^{\frac{s}{2}\Delta}$ on polynomials.

Another benefit of working with polynomials is that each polynomial f on \mathbb{R}^k already has an analytic continuation $f_{\mathbb{C}}$ to a unique polynomial on $(\mathbb{R}^k)_{\mathbb{C}} = \mathbb{C}^k$. The analytic continuation commutes with the Laplacian and hence heat operator. To avoid confusion on this point, let us introduce the following notation.

Notation 4.8. For $h \in C^\infty(\mathbb{C}^k)$, denote by $\Delta_{\mathbb{R}^k} h$ the Laplacian in the real variables:

$$(\Delta_{\mathbb{R}^k} h)(x + iy) = \sum_{j=1}^k \frac{\partial^2 h}{\partial x_j^2}(x + iy).$$

In particular, even if h is holomorphic, $\Delta_{\mathbb{R}^k} h \neq 0$ in general.

Proposition 4.9. Let f be a polynomial on \mathbb{R}^k of degree $\leq 2d$. Then

$$B_{s,\tau} f = \sum_{n=0}^d \frac{(\tau/2)^n}{n!} (\Delta^n f)_{\mathbb{C}} = \sum_{n=0}^d \frac{(\tau/2)^n}{n!} (\Delta_{\mathbb{R}^k})^n f_{\mathbb{C}}.$$

In particular, $B_{s,\tau} f(z)$ is a holomorphic polynomial in τ and z .

Proof. From definition (1.5), the restriction $B_{s,\tau} f|_{\mathbb{R}^k}$ can be expressed (by changing variables) as

$$B_{s,\tau} f|_{\mathbb{R}^k}(x) = \int_{\mathbb{R}^k} \rho_{\mathbb{C}}(\tau, x - v) f(v) dv = \int_{\mathbb{R}^k} \rho_{\mathbb{C}}(\tau, v) f(x - v) dv. \quad (4.6)$$

It follows that, for all $z \in \mathbb{C}^k$,

$$B_{s,\tau} f(z) = \int_{\mathbb{R}^k} \rho_{\mathbb{C}}(\tau, v) f_{\mathbb{C}}(z - v) dv. \quad (4.7)$$

(This follows since the right-hand-side is also holomorphic, as can be easily proven using standard differentiation-under-the-integral techniques for continuous functions, and uniqueness of analytic continuation.) What's more: taking $\tau = t \in \mathbb{R}$, since $\rho_{\mathbb{C}}(t, v) = \rho_t(v)$ for $v \in \mathbb{R}$ and this is an approximate identity (cf. Theorem 2.30(5)), it also follows from a similar analytic continuation argument that, for all $z \in \mathbb{C}^k$,

$$(B_{s,t} f)(z) = \int_{\mathbb{R}^k} \rho_t(v) f_{\mathbb{C}}(z - v) dv \rightarrow f_{\mathbb{C}}(z) \quad \text{as } t \downarrow 0. \quad (4.8)$$

(Theorem 2.30(5) technically only applies with $f \in C_c(\mathbb{R}^k)$, but we can use a bump function in a neighborhood of $z \in \mathbb{R}$ to apply it in this case.)

Now, the usual differentiation formula $\frac{d}{d\tau} \sqrt{\tau} = \frac{1}{2\sqrt{\tau}}$ holds with any branch of the square root function on \mathbb{C} , and so the same elementary calculus argument which shows ρ_t satisfies the heat equation shows that

$$\frac{\partial}{\partial \tau} \rho_{\mathbb{C}}(\tau, z) = \frac{1}{2} \Delta_{\mathbb{R}^k} \rho_{\mathbb{C}}(\tau, z). \quad (4.9)$$

It follows, again using standard differentiation-under-the-integral techniques for continuous functions, that

$$\frac{\partial}{\partial \tau} (B_{s,\tau} f)(z) = \int_{\mathbb{R}^k} \rho_{\mathbb{C}}(\tau, z - v) f(v) dv = \int_{\mathbb{R}^k} \Delta_{\mathbb{R}^k} \rho_{\mathbb{C}}(\tau, z - v) f(v) dv$$

where the Laplacian $\Delta_{\mathbb{R}^k}$ is applied to the second variable of $\rho_{\mathbb{C}}$. A double integration by parts (justified by the Gaussian tails of the real and imaginary parts of $\rho_{\mathbb{C}}(\tau, \cdot)$) now gives

$$\frac{\partial}{\partial \tau} (B_{s,\tau} f)(z) = \frac{1}{2} \int_{\mathbb{R}^k} \rho_{\mathbb{C}}(\tau, z - v) (\Delta f)(v) dv = \frac{1}{2} B_{s,\tau} (\Delta f)(z). \quad (4.10)$$

Now, $\Delta^n f \equiv 0$ for $n > d$; it thus follows from (4.10) that $\frac{\partial^n}{\partial \tau^n} B_{s,\tau} f \equiv 0$ for $n > d$. Therefore, using Taylor's theorem (with integral remainder term) to expand about a point

$t > 0$, we have

$$B_{s,\tau}f = \sum_{n=0}^d \frac{1}{n!} \frac{\partial^n}{\partial \tau^n} B_{s,\tau}f \Big|_{\tau=t} (\tau - t)^n = \sum_{n=0}^d \frac{(\tau - t)^n}{2^n n!} B_{s,t}(\Delta^n f), \quad \text{for } \operatorname{Re} \tau > 0.$$

We now take the limit $t \downarrow 0$, and apply (4.8) to each term in this finite sum (to the polynomials $\Delta^n f$) to find that

$$B_{s,\tau}f = \sum_{n=0}^d \frac{\tau^n}{2^n n!} (\Delta^n f)_{\mathbb{C}}$$

justifying the first equality in the statement proposition. Induction in the immediately verified identity $(\Delta f)_{\mathbb{C}} = \Delta_{\mathbb{R}^k} f_{\mathbb{C}}$ then justifies the second statement, concluding the proof. \square

Appealing to Proposition 2.45 and (2.20), Proposition 4.9 says that for polynomials f ,

$$B_{s,\tau}f = (e^{\frac{\tau}{2}\Delta} f)_{\mathbb{C}} = e^{\frac{\tau}{2}\Delta_{\mathbb{R}^k}} f_{\mathbb{C}}. \quad (4.11)$$

We will therefore be able to appeal to the power series definition to rigorously justify the preceding computations.

It will be convenient to express our Laplacians in terms of complex vector fields.

Notation 4.10. For $1 \leq j \leq k$,

$$\frac{\partial}{\partial z_j} = \frac{1}{2} \left(\frac{\partial}{\partial x} - i \frac{\partial}{\partial y} \right) \quad \text{and} \quad \frac{\partial}{\partial \bar{z}_j} = \frac{1}{2} \left(\frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right).$$

Define the following operators on $C^2(\mathbb{C}^k)$:

$$\partial^2 = \sum_{j=1}^k \frac{\partial^2}{\partial z_j^2} \quad \text{and} \quad \bar{\partial}^2 = \sum_{j=1}^k \frac{\partial^2}{\partial \bar{z}_j^2}.$$

Fundamentally, the proof of the Segal–Bargmann isometry boils down to the following elementary computation.

Lemma 4.11. For any $\tau \in \mathbb{C}_+$ and $s > 0$,

$$s\Delta_{\mathbb{R}^k} = \Delta_{s,\tau} + \tau\partial^2 + \bar{\tau}\bar{\partial}^2.$$

The proof of Lemma 4.11 is left to the reader; the details of the computation in the compact group case can be found in the proof of Lemma 4.25 below.

Theorem 4.12. Let f be a polynomial on \mathbb{R}^k . Then for any $\tau \in \mathbb{C}_+$ and $s > 0$ satisfying condition (1.6),

$$\|B_{s,\tau}f\|_{L^2(\mathbb{C}^k, \mu_{s,\tau})} = \|f\|_{L^2(\mathbb{R}^k, \rho_s)}.$$

Proof. By Proposition 4.9, $F = B_{s,\tau}f$ is a polynomial on \mathbb{C}^k . The original polynomial f has an analytic continuation $f_{\mathbb{C}}$ to \mathbb{C}^k as well. By Proposition 2.45,

$$\|F\|_{L^2(\mathbb{C}^k, \mu_{s,\tau})}^2 = \left(e^{\frac{1}{2}\Delta_{s,\tau}} |F|^2 \right) (0).$$

Now, $F = B_{s,\tau}f = e^{\frac{\tau}{2}\Delta_{\mathbb{R}^k}} f_{\mathbb{C}}$ by (4.11). By the Cauchy–Riemann equations, $\frac{\partial}{\partial \bar{z}_j} f_{\mathbb{C}} = \frac{\partial}{\partial x_j} f_{\mathbb{C}}$; thus $\partial^2 f_{\mathbb{C}} = \Delta_{\mathbb{R}^k} f_{\mathbb{C}}$. Therefore, from the power series definition, we have

$$F = e^{\frac{\tau}{2}\partial^2} f_{\mathbb{C}}, \quad \text{and taking complex conjugates,} \quad \bar{F} = e^{\frac{\bar{\tau}}{2}\bar{\partial}^2} \bar{f}_{\mathbb{C}} \quad (4.12)$$

Since $f_{\mathbb{C}}$ is holomorphic, $\bar{\partial}^2 f_{\mathbb{C}} = 0$; similarly $\partial^2 \bar{f}_{\mathbb{C}} = 0$. It therefore follows from (4.12) that

$$\begin{aligned} |F|^2 &= F\bar{F} = \left(e^{\frac{\tau}{2}\partial^2} f_{\mathbb{C}}\right) \left(e^{\frac{\bar{\tau}}{2}\bar{\partial}^2} \bar{f}_{\mathbb{C}}\right) \\ &= e^{(\frac{\tau}{2}\partial^2 + \frac{\bar{\tau}}{2}\bar{\partial}^2)}(f_{\mathbb{C}}\bar{f}_{\mathbb{C}}) = e^{(\frac{\tau}{2}\partial^2 + \frac{\bar{\tau}}{2}\bar{\partial}^2)}|f_{\mathbb{C}}|^2. \end{aligned} \quad (4.13)$$

Here we have used the fact that ∂^2 and $\bar{\partial}^2$ commute on $C^\infty(\mathbb{C}^k)$.

Combining (4.13) with (2.27), and using the fact that $\Delta_{s,\tau}$ commutes with both ∂^2 and $\bar{\partial}^2$ on $C^\infty(\mathbb{C}^k)$, we have

$$\|F\|_{L^2(\mathbb{C}^k, \mu_{s,\tau})}^2 = \left(e^{\frac{1}{2}(\Delta_{s,\tau} + \frac{\tau}{2}\partial^2 + \frac{\bar{\tau}}{2}\bar{\partial}^2)}|f_{\mathbb{C}}|^2\right)(0).$$

By Lemma 4.11, this becomes

$$\|B_{s,\tau} f\|_{L^2(\mathbb{C}^k, \mu_{s,\tau})}^2 = \|F\|_{L^2(\mathbb{C}^k, \mu_{s,\tau})}^2 = (e^{\frac{s}{2}\Delta_{\mathbb{R}^k}}|f_{\mathbb{C}}|^2)(0). \quad (4.14)$$

Since $|f_{\mathbb{C}}|^2|_{\mathbb{R}^k} = |f|^2$, and since $0 \in \mathbb{R}^k$ and $\Delta_{\mathbb{R}^k}$ leaves $C^\infty(\mathbb{R}^k)$ invariant, it follows that this is equal to $(e^{\frac{s}{2}\Delta_{\mathbb{R}^k}}|f|^2)(0)$, and by Proposition 2.45, that is equal to $\|f\|_{L^2(\mathbb{R}^k, \rho_s)}^2$, concluding the proof. \square

Finally, let's address the range of the transform on polynomials.

Lemma 4.13. *Every holomorphic polynomial F on \mathbb{C}^k has the form $F = B_{s,\tau} f$ for some polynomial f on \mathbb{R}^k .*

Proof. Let F be a holomorphic polynomial on \mathbb{C}^k , and denote $F|_{\mathbb{R}^k} = h$. Define $f = e^{-\frac{\tau}{2}\Delta}h$. Then from (4.11),

$$B_{s,\tau} f = (e^{\frac{\tau}{2}\Delta} f)_{\mathbb{C}} = (e^{\frac{\tau}{2}\Delta} e^{-\frac{\tau}{2}\Delta} h)_{\mathbb{C}} = h_{\mathbb{C}} = F.$$

\square

4.1.5. Extension to all of $L^2(\mathbb{R}^k, \rho_s)$. In Section 4.1.4 we showed that the restriction of $B_{s,\tau}$ to polynomials is an isometry (Theorem 4.12). Since polynomials are dense in $L^2(\mathbb{R}^k, \rho_s)$ (Proposition 4.6), if we knew *a priori* that $B_{s,\tau}$ were bounded from $L^2(\mathbb{R}^k, \rho_s)$ into $\mathcal{H}L^2(\mathbb{C}^k, \mu_{s,\tau})$, then we could now immediately conclude that it is an isometry on the full L^2 space. Since we do not know it is bounded, we will take an intermediate step.

Proposition 4.14. *Let $\tau \in \mathbb{C}_+$ and $s > 0$ satisfy condition (1.6). There is a unique isometric isomorphism $\overline{B}_{s,\tau}: L^2(\mathbb{R}^k, \rho_s) \rightarrow \mathcal{H}L^2(\mathbb{C}^k, \mu_{s,\tau})$ such that $\overline{B}_{s,\tau}$ agrees with $B_{s,\tau}$ on polynomials.*

Proof. We define $\overline{B}_{s,\tau}$ as follows: let $f \in L^2(\mathbb{R}^k, \rho_s)$, and let p_n be any sequence of polynomials such that $\|p_n - f\|_{L^2(\mathbb{R}^k, \rho_s)} \rightarrow 0$ as $n \rightarrow \infty$. By the isometry property of Theorem 4.12, the sequence $B_{s,\tau} p_n$ is Cauchy in $L^2(\mathbb{C}^k, \mu_{s,\tau})$:

$$\|B_{s,\tau} p_n - B_{s,\tau} p_m\|_{L^2(\mathbb{C}^k, \mu_{s,\tau})} = \|p_n - p_m\|_{L^2(\mathbb{R}^k, \rho_s)} \rightarrow 0 \text{ as } m, n \rightarrow \infty.$$

Since the terms $B_{s,\tau} p_n$ are holomorphic polynomials (Proposition 4.9), they are all in the subspace $\mathcal{H}L^2(\mathbb{C}^k, \mu_{s,\tau})$, which is a closed subspace of $L^2(\mathbb{C}^k, \mu_{s,\tau})$ (Proposition 4.7) and is therefore Cauchy complete. Hence, $B_{s,\tau} p_n$ has an L^2 -limit $F \in \mathcal{H}L^2(\mathbb{C}^k, \mu_{s,\tau})$. We define $\overline{B}_{s,\tau} f := F$.

This is well-defined: if q_n is a different sequence of polynomials approximating f in $L^2(\mathbb{R}^k, \rho_s)$, then using the isometry property of Theorem 4.12,

$$\begin{aligned} \|B_{s,\tau}p_n - B_{s,\tau}q_n\|_{L^2(\mathbb{C}^k, \mu_{s,\tau})} &= \|p_n - q_n\|_{L^2(\mathbb{R}^k, \rho_s)} \\ &\leq \|p_n - f\|_{L^2(\mathbb{R}^k, \rho_s)} + \|q_n - f\|_{L^2(\mathbb{R}^k, \rho_s)} \rightarrow 0. \end{aligned}$$

It is also evidently the unique continuous extension of $B_{s,\tau}$ on polynomials. That it is a linear isometry follows easily from the definition and the density of polynomials together with Theorem 4.12. Finally, it is surjective since the range of $B_{s,\tau}$ on polynomials is all holomorphic polynomials (Lemma 4.13), and these are dense in $\mathcal{H}L^2(\mathbb{C}^k, \mu_{s,\tau})$ (Proposition 4.6). This concludes the proof. \square

Whence, in order to complete the proof of Theorem 1.3 in the Euclidean setting, it behoves us to show that this unique extension $\overline{B}_{s,\tau}$ is actually equal to $B_{s,\tau}$. That is our final result.

Proof of Theorem 1.2 in the Euclidean Setting. Let $f \in L^2(\mathbb{R}^k, \rho_s)$. Fix any sequence of polynomials p_n with $\|p_n - f\|_{L^2(\mathbb{R}^k, \rho_s)} \rightarrow 0$. By definition, $\overline{B}_{s,\tau}f$ is the $L^2(\mathbb{C}^k, \mu_{s,\tau})$ -limit of $B_{s,\tau}p_n$. Now, using the pointwise bound of Proposition 4.1 and Remark 4.5, we have for each $z \in \mathbb{C}^k$

$$|(B_{s,\tau}p_n)(z) - (B_{s,\tau}p_m)(z)| \leq \left(\frac{s}{2\alpha}\right)^{k/2} \mu_{s,\tau}(z)^{-1/2} \|p_n - p_m\|_{L^2(\mathbb{R}^k, \rho_s)}$$

and this tends to 0 as $n, m \rightarrow \infty$. Thus $B_{s,\tau}p_n(z)$ is a Cauchy sequence for each z , and it follows $B_{s,\tau}p_n$ converges pointwise on \mathbb{C}^k . Since it has $B_{s,\tau}f$ as its L^2 -limit, this must also be the pointwise limit almost everywhere; since the functions $B_{s,\tau}f$ and $\overline{B}_{s,\tau}f$ are both holomorphic, they are actually equal everywhere. Thus, we conclude that $\overline{B}_{s,\tau}f = B_{s,\tau}f$ for all $f \in L^2(\mathbb{R}^k, \rho_s)$.

Proposition 4.14 shows that $B_{s,\tau} = \overline{B}_{s,\tau}$ has all the properties claimed in Theorem 1.3, and this concludes the proof. \square

4.1.6. Second Proof: Change of Variables. In this section, we briefly outline an alternate proof of Theorem 1.3 in the Euclidean setting, following only Section 4.1.1 (the complexification of the heat kernel). We will show how the statement of Theorem 1.3 can, by a change of variables, be reduced to the already known case for τ real, cf. [8].

Notation 4.15. In this section, we adopt the notation that, for vectors $z \in \mathbb{C}^k$,

$$z^2 := z \cdot z = \sum_{j=1}^k z_j^2.$$

The Segal–Bargmann transform is (nominally, without going through the justification of Section 4.1.2) given by

$$B_{s,\tau}f(z) = \int_{\mathbb{R}^k} \rho_{\mathbb{C}}(\tau, z-v)f(v) dv = (\sqrt{2\pi\tau})^{-k} \int_{\mathbb{R}^k} e^{-(z-v)^2/2\tau} f(v) dv.$$

Define $\sigma = |\tau|^2/\operatorname{Re} \tau = (t^2 + u^2)/t$, and let $\beta = \bar{\tau}/\operatorname{Re} \tau = 1 - iu/t$. The complex Gaussian density in the complexified heat kernel is then equal to

$$e^{-(z-v)^2/2\tau} = \exp\left\{-\frac{\bar{\tau}}{2|\tau|^2}(z-v)^2\right\} = \exp\left\{-\frac{\beta}{2\sigma}(z-v)^2\right\}.$$

We expand the quadratic form (sans $1/2\sigma$) as follows:

$$\beta(z-v)^2 = (\beta - \beta^2)z^2 + (\beta z - v)^2 + i\operatorname{Im}\beta v^2$$

where we have used the fact that $\operatorname{Re} \beta = 1$. Thus, the transform can be nominally rewritten as

$$B_{s,\tau}f(z) = (\sqrt{2\pi\tau})^{-k} e^{-(\beta-\beta^2)z^2/2\sigma} \int_{\mathbb{R}^k} e^{-(\beta z-v)^2/2\sigma} e^{i(u/t)v^2/2\sigma} f(v) dv.$$

Letting $w = \beta z$, we have

$$B_{s,\tau}f(w/\beta) = (\sqrt{2\pi\tau})^{-k} e^{-(1/\beta-1)w^2/2\sigma} \int_{\mathbb{R}^k} e^{-(w-v)^2/2\sigma} e^{i(u/t)v^2/2\sigma} f(v) dv. \quad (4.15)$$

Now, fix $s > 0$ satisfying condition (1.6), meaning that $s > \sigma/2$. By [8, Theorem 5.3] (which is the $\tau \in \mathbb{R}$ version of our main Theorem 1.3), the real-time Segal–Bargmann transform

$$B_{s,\sigma}f(w) = (2\pi\sigma)^{-k/2} \int_{\mathbb{R}^k} e^{-(w-v)^2/2\sigma} f(v) dv$$

is an isometric isomorphism from $L^2(\mathbb{R}^k, \rho_s)$ onto $\mathcal{H}L^2(\mathbb{C}^k, \mu_{s,\sigma})$, precisely under the given condition $s > \sigma/2$. Referring to (4.15), note that

$$B_{s,\tau}f(w/\beta) = (\sigma/\tau)^{k/2} e^{-(1/\beta-1)w^2/2\sigma} B_{s,\sigma}\check{f}(w) \quad (4.16)$$

where $\check{f}(v) = e^{i(u/t)v^2/2\sigma} f(v)$, which is just a complex phase times f . Thus, using (4.16), we have

$$\begin{aligned} \|f\|_{L^2(\mathbb{R}^k, \rho_s)}^2 &= \|\check{f}\|_{L^2(\mathbb{R}^k, \rho_s)}^2 = \|B_{s,\sigma}\check{f}\|_{L^2(\mathbb{C}^k, \mu_{s,\sigma})}^2 \\ &= (|\tau|/\sigma)^k \int_{\mathbb{C}^k} |e^{(1/\beta-1)w^2/\sigma}| |B_{s,\tau}f(w/\beta)|^2 \mu_{s,\sigma}(w) dw \\ &= (|\tau|/\sigma)^k \int_{\mathbb{C}^k} |e^{(\beta-\beta^2)z^2/\sigma}| |B_{s,\tau}f(z)|^2 |\mu_{s,\sigma}(\beta z)| |\beta|^{2k} dz. \end{aligned}$$

Note that $|\tau||\beta|/\sigma = 1$. This shows that $B_{s,\tau}$ is an isometry from $L^2(\mathbb{R}^k, \rho_s)$ into the holomorphic L^2 space of the measure whose density is

$$|e^{(\beta-\beta^2)z^2/\sigma}| |\beta|^k |\mu_{s,\sigma}(\beta z)| \quad (4.17)$$

where $\mu_{s,\sigma}(x+iy) = (\pi\sigma)^{-k/2} (\pi(2s-\sigma))^{-k/2} e^{-x^2/(2s-\sigma)} e^{-y^2/\sigma}$ (cf. Remark 4.4). To be clear, we then have

$$|\mu_{s,\sigma}(\beta z)| = (\pi\sigma)^{-k/2} (\pi(2s-\sigma))^{-k/2} \exp \left\{ -\frac{[\operatorname{Re}(\beta z)]^2}{2s-\sigma} - \frac{[\operatorname{Im}(\beta z)]^2}{\sigma} \right\}.$$

Note that $\operatorname{Re}(\beta z) = x + \frac{u}{t}y$ and $\operatorname{Im}(\beta z) = y - \frac{u}{t}x$. We also compute that

$$\operatorname{Re} \left[\frac{\beta - \beta^2}{\sigma} z^2 \right] = \frac{u}{|\tau|^2} (|x|^2 - |y|^2) - \frac{2tu}{|\tau|^2} x \cdot y.$$

Hence, (4.17) becomes

$$|\beta|^k \pi^{-k} (\sigma(2s-\sigma))^{-k/2} \exp \left\{ \frac{u}{|\tau|^2} (|x|^2 - |y|^2) - \frac{2tu}{|\tau|^2} x \cdot y - \frac{|x + \frac{u}{t}y|^2}{2s-\sigma} - \frac{|y - \frac{u}{t}x|^2}{\sigma} \right\}.$$

It is now a tedious but mundane calculation to show that this is equal to $\mu_{s,\tau}(z)$ (cf. (4.5)), proving the desired isometry. The surjectivity follows in a similar backtracking fashion.

4.2. The Compact Group Case. In this section, we work on a fixed compact Lie group K , with a given $\text{Ad}(K)$ -invariant inner product on its Lie algebra \mathfrak{k} . There is no “closed form” formula for the heat kernel in this context. If we wished to follow precisely the outline of Section 4.1, our first step would be to prove that the heat kernel $(\rho_t(x))_{t>0, x \in K}$ on K has an analytic continuation $(\rho_{\mathbb{C}}(\tau, z))_{\tau \in \mathbb{C}_+, z \in K_{\mathbb{C}}}$. This is possible to do, following exactly the proof in [16, Section 4].

The idea is to use *Stein’s character formula* for the heat kernel. Recall that \hat{K} denotes the set of irreducible representations (π, V_{π}) of K . The Casimir invariant C_{π} commutes with the representation π (cf. (2.23), Remark 2.42), and so by Schur’s lemma there is a constant $\lambda_{\pi} \in \mathbb{C}$ so that $C_{\pi} = -\lambda_{\pi}I$. (The choice of sign is because this makes $\lambda_{\pi} \geq 0$: since K is compact, π is equivalent to a unitary representation, and thus $\pi_*(X)$ is skew-Hermitian for each $X \in \mathfrak{k}$. It follows that $C_{\pi} = \sum_j \pi_*(X_j)^2$ is negative semidefinite.) Then the heat kernel has the following explicit series representation:

$$\rho_t(x) = \sum_{\pi \in \hat{K}} \dim V_{\pi} e^{-\lambda_{\pi} t/2} \chi_{\pi}(x) \quad (4.18)$$

where $\chi_{\pi}(x) = \text{Tr}(\pi(x)) = f_{\pi, I}(x)$ is the character of π , cf. Definition 2.36. Nominally, the analytic continuation of $\rho_t(x)$ can then be defined by substituting $\tau \in \mathbb{C}_+$ for t and extending $\chi_{\pi}(z)$ to $z \in K_{\mathbb{C}}$ (via the universal property of the complexification, on p. 8). The work is in proving this series converges uniformly on compact subsets of $K_{\mathbb{C}}$. In fact, the estimates in [16, Lemmas 6 & 7] suffice for this task in the complex time case just as well (since they all involve a modulus of an exponential, ergo only the real part of the time).

From here we would have to prove (crude) pointwise bounds for the transform as in Proposition 4.1. While this is possible, we will take a different approach to the transform and analytically continued heat kernel in this setting. We define an ostensibly different transform $M_{s, \tau}$ on matrix entries, using (2.20) and mimicking the approach of Section 4.1.4. We then prove $M_{s, \tau}$ extends to an isometric isomorphism, following closely the proof in Section 4.1.4; this is all done in Section 4.2.2. In order to import the isometry proof to this non-abelian group setting, we must first prove that the analogs of the operators ∂^2 and $\bar{\partial}^2$ commute with $\Delta_{\mathfrak{k}}$ and $\Delta_{s, \tau}$, which is done in Section 4.2.1. Then we *apply the transform $M_{s, \tau}$ to the heat kernel* to show that it has an analytic continuation, and finally prove that the transform is given by integration against this complexified heat kernel as in (1.5), in Section 4.2.3. Finally, Section 4.2.4 is devoted to the proof of Theorem 1.4 (the $s \rightarrow \infty$ limit version of the transform).

4.2.1. Complex Vector Fields and Commutation Relations. We would now like to emulate the proof of the Segal–Bargmann isometry for the \mathbb{R}^k case given in Section 4.1.4, cf. Theorem 4.12. To that end, we must introduce the complex vector fields generalizing the complex derivatives $\frac{\partial}{\partial z_j}$ and $\frac{\partial}{\partial \bar{z}_j}$ in the Euclidean context.

Definition 4.16. *Let G be a complex Lie group with Lie algebra \mathfrak{g} . Let $V \in \mathfrak{g}$. The **holomorphic and antiholomorphic vector fields** associated to V are complex vector fields ∂_V and $\bar{\partial}_V$ on G defined by*

$$\partial_V \equiv \frac{1}{2} \left(\tilde{V} - i\widetilde{JV} \right) \quad \text{and} \quad \bar{\partial}_V \equiv \frac{1}{2} \left(\tilde{V} + i\widetilde{JV} \right). \quad (4.19)$$

In the special case $G = \mathbb{C}^d$, if $V = \frac{\partial}{\partial x^j}$ then $\partial_V = \frac{\partial}{\partial z^j}$ and $\bar{\partial}_V = \frac{\partial}{\partial \bar{z}^j}$. (The reader is warned, therefore, that the notation is somewhat counterintuitive when comparing to the

classical context.) We can then recover V and JV as follows:

$$\tilde{V} = \partial_V + \bar{\partial}_V, \quad \text{and} \quad \widetilde{JV} = i(\partial_V - \bar{\partial}_V).$$

The Cauchy-Riemann equations in local coordinates give the following.

Lemma 4.17. *If $U \subseteq G$ is open and $f \in \mathcal{H}(U)$, then for any $V \in \mathfrak{g}$, it follows that $\partial_V f = \tilde{V}f$, while $\bar{\partial}_V f = 0$. In addition, $\bar{\partial}_V \bar{f} = \overline{\partial_V f}$.*

Remark 4.18. Note that ∂_V and $\bar{\partial}_V$ are left-invariant vector fields on G . Nevertheless, there is no element $Z \in \mathfrak{g}$ such that $\partial_V = \tilde{Z}$. Indeed, by Lemma 4.17, if such a Z exists then $\tilde{V}f = \tilde{Z}f$ for all locally holomorphic functions f on G . Taking f to be the coordinate functions in a holomorphic chart shows that $\tilde{V} = \tilde{Z}$, and so $V = Z$. It follows that $\tilde{V} = \partial_V = \frac{1}{2}(\tilde{V} - i\widetilde{JV})$, meaning that $\tilde{V} = -i\widetilde{JV}$, which implies that $\bar{\partial}_V = 0$. This is, of course, impossible: the complex conjugates of any coordinate functions are antiholomorphic and therefore not in the kernel of $\bar{\partial}_V$ for some $V \in \mathfrak{g}$.

This points out the important fact that the identification between \mathfrak{g} and the left-invariant vector fields on G is for *real* vector fields only, even if \mathfrak{g} is complex. This is why we need the global section J , as the next lemma demonstrates.

Lemma 4.19. *If $X, V \in \mathfrak{g}$, then*

$$[\partial_V, \widetilde{JX}] = i[\partial_V, \tilde{X}], \quad \text{and} \quad [\bar{\partial}_V, \widetilde{JX}] = -i[\bar{\partial}_V, \tilde{X}].$$

Proof. By Corollary 2.3, for any $W_1, W_2 \in \mathfrak{g}$, $[JW_1, W_2] = J[W_1, W_2] = [W_1, JW_2]$. We can then compute from the definition that

$$[\partial_V, \widetilde{JX}] = \frac{1}{2}[\tilde{V} - i\widetilde{JV}, \widetilde{JX}] = \frac{1}{2}[\widetilde{JV} - i\widetilde{J\tilde{V}}, \tilde{X}] = \frac{1}{2}[\widetilde{JV} + i\tilde{V}, \tilde{X}] = i[\partial_V, \tilde{X}].$$

The calculation for $\bar{\partial}_V$ is similar. \square

We now specialize to the case $G = K_{\mathbb{C}}$ for a compact Lie group K , and introduce the analogs of the operators ∂^2 and $\bar{\partial}^2$ from Notation 4.10.

Definition 4.20. *Fix an orthonormal basis $\{X_1, \dots, X_d\}$ for \mathfrak{k} , and let $\partial_j := \partial_{X_j}$ as in (4.19). Then set*

$$\partial^2 \equiv \sum_{j=1}^d \partial_j^2, \quad \text{and} \quad \bar{\partial}^2 \equiv \sum_{j=1}^d \bar{\partial}_j^2. \quad (4.20)$$

Lemma 4.21. *The operators ∂^2 and $\bar{\partial}^2$ are well-defined, independent of basis.*

Proof. Each of these operators has the form $\sum_{j=1}^d T(X_j, X_j)$ where $T: \mathfrak{k} \times \mathfrak{k} \rightarrow \mathfrak{U}(K_{\mathbb{C}})$ is a real bilinear form into the universal enveloping algebra of left-invariant differential operators on $K_{\mathbb{C}}$. By the universal property of tensor products, there is a unique linear map $\hat{T}: \mathfrak{k} \otimes \mathfrak{k} \rightarrow \mathfrak{U}(K_{\mathbb{C}})$ such that $T(X, Y) = \hat{T}(X \otimes Y)$, and so the operators each have the form $\hat{T}(\sum_{j=1}^d X_j \otimes X_j)$. Now, there is a natural identification $\Psi: \mathfrak{k} \otimes \mathfrak{k} \rightarrow \text{End}(\mathfrak{k})$ depending on the inner product but not the specific basis, given by $\Psi(X \otimes Y)(V) = \langle Y, V \rangle_{\mathfrak{k}} X$. (In standard physics “bra-ket” notation, this is written as $\Psi(X \otimes Y) = |X\rangle\langle Y|$.) Under this identification, it is easy to check that $\Psi(\sum_{j=1}^d X_j \otimes X_j) = I$ is the identity map, which is clearly basis independent. Hence each of the operators has the form $\hat{T}(\Psi^{-1}(I))$, and therefore they are basis independent. \square

Lemma 4.22. *For all $f \in C^\infty(K_{\mathbb{C}})$ and all $k \in K$,*

$$\partial^2(f \circ R_k) = (\partial^2 f) \circ R_k, \quad \text{and} \quad \bar{\partial}^2(f \circ R_k) = (\bar{\partial}^2 f) \circ R_k. \quad (4.21)$$

The proof of Lemma 4.22 is very similar to the proof of Lemma 2.24, and is left to the reader.

This brings us to the main commutator result of this section.

Proposition 4.23. *For any $A \in \mathfrak{k}_{\mathbb{C}}$,*

$$[\partial^2, \tilde{A}] = [\bar{\partial}^2, \tilde{A}] = 0.$$

Proof. As any $A \in \mathfrak{k}_{\mathbb{C}}$ has the form $A = V + JW$ for some $V, W \in \mathfrak{k}$, it suffices by linearity to prove that ∂^2 and $\bar{\partial}^2$ commute with \tilde{V} and \widetilde{JW} for any $V \in \mathfrak{k}$. For the former statement, apply Lemma 4.22 with $k = e^{tV}$, and differentiate at $t = 0$ to yield the result. For the second statement, we employ Lemma 4.19 and compute as follows.

$$\begin{aligned} [\partial^2, \widetilde{JW}] &= \sum_{j=1}^d [\partial_j \partial_j, \widetilde{JW}] = \sum_{j=1}^d \left(\partial_j [\partial_j, \widetilde{JW}] + [\partial_j, \widetilde{JW}] \partial_j \right) \\ &= i \sum_{j=1}^d \left(\partial_j [\partial_j, \tilde{V}] + [\partial_j, \tilde{V}] \partial_j \right) = i \sum_{j=1}^d [\partial_j \partial_j, \tilde{V}] = i [\partial^2, \tilde{V}] \end{aligned}$$

and we already showed that $[\partial^2, \tilde{V}] = 0$. A similar calculation proves the result for $\bar{\partial}^2$. \square

Corollary 4.24. *The operators ∂^2 , $\bar{\partial}^2$, $\Delta_{\mathfrak{k}}$, and $\Delta_{s,\tau}$ all mutually commute.*

Proof. Since $\Delta_{\mathfrak{k}}$ and $\Delta_{s,\tau}$ are linear combinations of squares of left-invariant vector fields on $K_{\mathbb{C}}$, Proposition 4.23 shows that they both commute with ∂^2 and $\bar{\partial}^2$. Similarly, letting $Y_j = JX_j$, since ∂_j^2 and $\bar{\partial}_j^2$ are linear combinations of \tilde{X}_j^2 , \tilde{Y}_j^2 , and $\tilde{X}_j \tilde{Y}_j = \tilde{Y}_j \tilde{X}_j$ (cf. 2.3), the commutator $[\partial^2, \bar{\partial}^2] = 0$ also follows from Proposition 4.23. Finally, $\Delta_{\mathfrak{k}}$ and $\Delta_{s,\tau}$ commute by Theorem 2.34(1) (cf. Remark 2.35). \square

The usefulness of the ∂^2 and $\bar{\partial}^2$ operators in the present context lies in the following exact analog of Lemma 4.11, whose proof we spell out to highlight what commutation relations are needed.

Lemma 4.25. *Let $\tau \in \mathbb{C}_+$ and $s > 0$ satisfy (1.6). Let $\Delta_{s,\tau}$ denote the $K_{\mathbb{C}}$ Laplacian of Definition 3.5, and let $\Delta_{\mathfrak{k}}$ denote the Laplacian of K acting on $C^\infty(K_{\mathbb{C}})$ as usual. Then*

$$s\Delta_{\mathfrak{k}} = \Delta_{s,\tau} + \tau\partial^2 + \bar{\tau}\bar{\partial}^2.$$

Proof. Fix an orthonormal basis $\{X_1, \dots, X_d\}$ of \mathfrak{k} . For ease of reading, let $Y_j = JX_j$. To begin, we compute that, for each j ,

$$\partial_j^2 + \bar{\partial}_j^2 = \frac{1}{4}(\tilde{X}_j - i\tilde{Y}_j)^2 + \frac{1}{4}(\tilde{X}_j + i\tilde{Y}_j)^2 = \frac{1}{2}(\tilde{X}_j^2 - \tilde{Y}_j^2), \quad (4.22)$$

$$\partial_j^2 - \bar{\partial}_j^2 = \frac{1}{4}(\tilde{X}_j - i\tilde{Y}_j)^2 - \frac{1}{4}(\tilde{X}_j + i\tilde{Y}_j)^2 = -i\tilde{X}_j \tilde{Y}_j \quad (4.23)$$

where we have used the fact that $[\tilde{X}_j, \tilde{Y}_j] = 0$ (cf. Corollary 2.3).

Now, let $\tau = t + iu$. Then for each j ,

$$\tau\partial_j^2 + \bar{\tau}\bar{\partial}_j^2 = t(\partial_j^2 + \bar{\partial}_j^2) + iu(\partial_j^2 - \bar{\partial}_j^2) = \frac{t}{2}(\tilde{X}_j^2 - \tilde{Y}_j^2) + u\tilde{X}_j \tilde{Y}_j.$$

Thus, we have

$$\left[\left(s - \frac{t}{2} \right) \tilde{X}_j^2 + \frac{t}{2} \tilde{Y}_j^2 - u\tilde{X}_j \tilde{Y}_j \right] + \tau\partial_j^2 + \bar{\tau}\bar{\partial}_j^2 = s\tilde{X}_j^2. \quad (4.24)$$

Summing (4.24) on j proves the lemma. \square

4.2.2. *The Transform $M_{s,\tau}$, and the Isomorphism $\overline{M}_{s,\tau}$.* We now define a transform $M_{s,\tau}$ directly by its action on matrix entries. Let $f_{\pi,A}$ be a matrix entry on K . By the universal property of complexifications, the representation (π, V_π) (which we assume is already on a complex space) has an analytic continuation $\pi_{\mathbb{C}}$ which is a holomorphic representation of $K_{\mathbb{C}}$ on V_π . Hence, the matrix entry $f_{\pi,A}$ has an analytic continuation as well,

$$(f_{\pi,A})_{\mathbb{C}}(g) = \text{Tr}(\pi_{\mathbb{C}}(g)A) = f_{\pi_{\mathbb{C}},A}(g), \quad g \in K_{\mathbb{C}}.$$

Definition 4.26. For $\tau \in \mathbb{C}_+$, define $M_{s,\tau}$ on matrix entries on K as

$$M_{s,\tau} f_{\pi,A} = e^{\frac{\tau}{2} \Delta_{\mathfrak{k}}} (f_{\pi,A})_{\mathbb{C}} = \sum_{n=0}^{\infty} \frac{(\tau/2)^n}{n!} (\Delta_{\mathfrak{k}})^n f_{\pi_{\mathbb{C}},A}.$$

To be clear: we are appealing to (2.20) here: the exponential is well-defined and yields the new matrix entry function

$$M_{s,\tau} f_{\pi,A} = f_{\pi_{\mathbb{C}}, e^{\tau C_{\pi_{\mathbb{C}}/2} \cdot} A} \quad (4.25)$$

which is a holomorphic matrix entry. Thus, $M_{s,\tau}$ maps matrix entries on K to holomorphic matrix entries on $K_{\mathbb{C}}$. And it is an isometry between the relevant L^2 -norms, as we now show.

Theorem 4.27. Let f be a matrix entry function on K . Then for any $\tau \in \mathbb{C}_+$ and $s > 0$ satisfying condition (1.6),

$$\|M_{s,\tau} f\|_{L^2(K_{\mathbb{C}}, \mu_{s,\tau})} = \|f\|_{L^2(K, \rho_s)}.$$

Moreover, every holomorphic matrix entry F on $K_{\mathbb{C}}$ has the form $F = M_{s,\tau} f$ for some matrix entry f on K .

Proof. The proof is essentially identical to the proof of Theorem 4.12 and Lemma 4.13. Let $f = f_{\pi,A}$, and set $F = M_{s,\tau} f = e^{\tau \Delta_{\mathfrak{k}}/2} f_{\mathbb{C}}$. By Lemma 4.17, $\partial_j f_{\mathbb{C}} = \tilde{X}_j f_{\mathbb{C}}$; squaring and summing, it follows that $\Delta_{\mathfrak{k}} f_{\mathbb{C}} = \partial^2 f_{\mathbb{C}}$, and similarly $\Delta_{\mathfrak{k}} \bar{f}_{\mathbb{C}} = \bar{\partial}^2 \bar{f}_{\mathbb{C}}$. It also follows from Lemma 4.17 that $\partial^2 \bar{f}_{\mathbb{C}} = \bar{\partial}^2 f_{\mathbb{C}} = 0$. Now appealing to the power series representations of (2.20), and using the commutation relations of Corollary 4.24, we deduce (4.13) again in this context:

$$|F|^2 = e^{(\frac{\tau}{2} \partial^2 + \frac{\tau}{2} \bar{\partial}^2)} |f_{\mathbb{C}}|^2.$$

Applying (2.27), again using Corollary 4.24, this yields

$$\|F\|_{L^2(K_{\mathbb{C}}, \mu_{s,\tau})}^2 = \left(e^{\frac{1}{2}(\Delta_{s,\tau} + \frac{\tau}{2} \partial^2 + \frac{\tau}{2} \bar{\partial}^2)} |f_{\mathbb{C}}|^2 \right) (e).$$

By Lemma 4.25, this becomes

$$\|M_{s,\tau} f\|_{L^2(K_{\mathbb{C}}, \mu_{s,\tau})}^2 = \|F\|_{L^2(K_{\mathbb{C}}, \mu_{s,\tau})}^2 = \left(e^{\frac{s}{2} \Delta_{\mathfrak{k}}} |f_{\mathbb{C}}|^2 \right) (e).$$

Since $|f_{\mathbb{C}}|^2|_K = |f|^2$, and since $e \in K$ and $\Delta_{\mathfrak{k}}$ leaves $C^\infty(K)$ invariant, it follows that this is equal to $(e^{\frac{s}{2} \Delta_{\mathfrak{k}}} |f|^2)(e)$, and by Proposition 2.45, that is equal to $\|f\|_{L^2(K, \rho_s)}^2$, completing the proof of the isometry.

The surjectivity proof is identical to the proof of Lemma 4.13, replacing $B_{s,\tau}$ with $M_{s,\tau}$, \mathbb{R}^k with K , and \mathbb{C}^k with $K_{\mathbb{C}}$. \square

We may now extend $M_{s,\tau}$ by continuity to an isometry on $L^2(K, \rho_s)$, following the proof of Proposition 4.14 nearly verbatim. First we need the $K_{\mathbb{C}}$ -analog of Proposition 4.7, which is (as in the \mathbb{C}^k context) a special case of [5, Theorem 3.2 & Corollary 3.3].

Proposition 4.28. *Let $\tau \in \mathbb{C}_+$ and $s > 0$ satisfy (1.6). The space $\mathcal{HL}^2(K_{\mathbb{C}}, \mu_{s,\tau})$ is a closed subspace of the Hilbert space $L^2(K_{\mathbb{C}}, \mu_{s,\tau})$. Moreover, for each $z \in K_{\mathbb{C}}$, the point evaluation linear functional given by $f \mapsto f(z)$ is continuous on $\mathcal{HL}^2(K_{\mathbb{C}}, \mu_{s,\tau})$.*

Proposition 4.29. *Let $\tau \in \mathbb{C}_+$ and $s > 0$ satisfy (1.6). There is a unique isometric isomorphism $\overline{M}_{s,\tau}: L^2(K, \rho_s) \rightarrow \mathcal{HL}^2(K_{\mathbb{C}}, \mu_{s,\tau})$ such that $\overline{M}_{s,\tau}$ agrees with $M_{s,\tau}$ on matrix entries.*

Proof. Since K is compact and ρ_s is continuous, we know that $0 < \min_{x \in K} \rho_s(x) \leq \max_{x \in K} \rho_s(x) < \infty$ and hence $L^2(K, \rho_s) = L^2(K)$ as vector spaces, with equivalent norms. Matrix entries are dense in $L^2(K)$ by the Peter–Weyl theorem, and hence they are dense in $L^2(K, \rho_s)$. We may then define $\overline{M}_{s,\tau} f$ as the L^2 -limit of $M_{s,\tau} f_n$ for any sequence of matrix entries f_n that approximate f in $L^2(K, \rho_s)$. The remainder of the proof of Proposition 4.14 now follows word for word to show that $\overline{M}_{s,\tau}$ is a well-defined isometry from $L^2(K, \rho_s)$ into $\mathcal{HL}^2(K_{\mathbb{C}}, \mu_{s,\tau})$. Since holomorphic matrix entries are dense in the codomain (cf. Theorem 3.11), the second statement of Theorem 4.27 thus shows that $\overline{M}_{s,\tau}$ is also surjective, concluding the proof. \square

Hence $\overline{M}_{s,\tau}$ is an isometric isomorphism from $L^2(K, \rho_s)$ onto $\mathcal{HL}^2(K_{\mathbb{C}}, \mu_{s,\tau})$. In particular, for any $f \in L^2(K, \rho_s)$, $(\overline{M}_{s,\tau} f)(z)$ is holomorphic in z . In fact, it is also holomorphic in τ , as we now proceed to show.

Remark 4.30. As noted above $L^2(K, \rho_s) = L^2(K)$ with equivalent norms for each $s > 0$. It is important to note, however, that the ρ_s -inner product on $L^2(K)$ varies with s .

Remark 4.31 (Pointwise bounds). If h is a left-invariant Riemannian metric on $K_{\mathbb{C}}$ and μ_t^h is the associated heat kernel on $K_{\mathbb{C}}$, it is shown in [6, Corollary 5.4 and Remark 5.5] that

$$|f(z)|^2 \leq \|f\|_{\mu_t^h}^2 e^{|z|_h^2/t} \text{ for all } z \in K_{\mathbb{C}} \text{ and } t > 0$$

where $|z|_h = d_h(z, e)$ is the distance from z to e in $K_{\mathbb{C}}$ relative to the length metric associated to h .

Now suppose that $h = h_\tau$ is a one parameter family of left invariant Riemannian metrics continuously varying with $\tau \in \mathbb{C}_+$ and let $|\xi|_\tau := \sqrt{h_\tau(\xi, \xi)}$ for all $\xi \in \mathfrak{k}_{\mathbb{C}}$. Then for each compact set $\mathcal{K} \subset \mathbb{C}_+$ and fixed point $\tau_0 \in \mathbb{C}_+$, there exists a constant $c_{\mathcal{K}} < \infty$ such that

$$c_{\mathcal{K}}^{-1} |\xi|_{\tau_0} \leq |\xi|_\tau \leq c_{\mathcal{K}} |\xi|_{\tau_0} \quad \forall \xi \in \mathfrak{k}_{\mathbb{C}} \text{ and } \tau \in \mathcal{K}.$$

It then easily follows that

$$c_{\mathcal{K}}^{-1} |z|_{h_{\tau_0}} \leq |z|_{h_\tau} \leq c_{\mathcal{K}} |z|_{h_{\tau_0}} \quad \forall z \in K_{\mathbb{C}} \text{ and } \tau \in \mathcal{K}$$

and in particular for all $f \in \mathcal{HL}^2(K_{\mathbb{C}}, \mu_t^{h_\tau})$ and $t > 0$, we have the following pointwise bounds,

$$|f(z)|^2 \leq \|f\|_{L^2(K_{\mathbb{C}}, \mu_t^{h_\tau})}^2 e^{c_{\mathcal{K}} |z|_{h_{\tau_0}}^2/t}. \quad (4.26)$$

Proposition 4.32. *Let $\tau \in \mathbb{C}_+$ and $s > 0$ satisfy (1.6). Given any $f \in L^2(K)$, the function $\mathbb{C}_+ \times K_{\mathbb{C}} \ni (\tau, z) \mapsto (\overline{M}_{s,\tau} f)(z)$ is holomorphic.*

Proof. We wish to show $(\tau, z) \mapsto (\overline{M}_{s,\tau} f)(z)$ is jointly holomorphic. This is true if $f = f_{\pi,A}$ is a matrix entry since by Definition 4.26 and (4.25),

$$(M_{s,\tau} f)(z) = f_{\pi_{\mathbb{C}}, e^{\tau C_{\pi_{\mathbb{C}}}/2} A}(z) = \text{Tr}(\pi_{\mathbb{C}}(z) e^{\frac{\tau}{2} C_{\pi_{\mathbb{C}}} A}).$$

As the representation $\pi_{\mathbb{C}}$ is finite dimensional, the matrix valued function $\tau \mapsto e^{\frac{\tau}{2}C_{\pi_{\mathbb{C}}}}$ is immediately seen to be holomorphic, and hence $(\tau, z) \mapsto (M_{s,\tau}f)(z)$ is jointly holomorphic.

For general $f \in L^2(K) = L^2(K, \rho_s)$, let f_n be any sequence of matrix entries on K with $\|f_n - f\|_{L^2(K, \rho_s)} \rightarrow 0$. Fix some $\tau_0 \in \mathbb{C}_+$ and some compact $\mathcal{K} \subset \mathbb{C}_+$. We now use the bound (4.26) (with $t = 1$ since the relevant heat kernel $\mu_{s,\tau}$ is evaluated at time 1, cf. Definition 3.6) and the isometry property of $\overline{M}_{s,\tau}$ in Proposition 4.29 to find that, for all $\tau \in \mathcal{K}$,

$$\begin{aligned} |(\overline{M}_{s,\tau}f)(z) - (M_{s,\tau}f_n)(z)| &\leq \|\overline{M}_{s,\tau}(f - f_n)\|_{L^2(K_{\mathbb{C}}, \mu_{s,\tau})}^2 e^{c_{\mathcal{K}}|z|_{\tau_0}^2} \\ &= \|f - f_n\|_{L^2(K, \rho_s)}^2 e^{c_{\mathcal{K}}|z|_{\tau_0}^2} \end{aligned}$$

and this shows $(M_{s,\tau}f_n)(z)$ converges to $(\overline{A}_{s,\tau}f)(z)$ locally uniformly in (τ, z) . (The constant $c_{\mathcal{K}}$ and the distance $|\cdot|_{\tau_0}$ also depend on $s > 0$, which is not varying in this proof.) As holomorphic functions are stable under locally uniform convergence, the proof is complete. \square

4.2.3. Proofs of Theorem 1.1 and 1.3. We can now show that the heat kernel, $\rho_t(x)$, has an analytic continuation in space and time, $\rho_{\mathbb{C}}(\tau, z)$.

Proof of Theorem 1.1 in the compact group case. For any $\epsilon > 0$, the heat kernel ρ_{ϵ} is a continuous function on the compact group K , and hence it is in $L^2(K, \rho_s)$ for any $s > 0$. Fix $\tau = t + iu \in \mathbb{C}_+$, let $0 < \epsilon < t$, and select $s > |\tau - \epsilon|^2/2(t - \epsilon)$ (i.e. so that condition (1.6) holds for the pair $s, \tau - \epsilon$). By Proposition 4.32, the function

$$\rho_{\mathbb{C}}(\tau, z) = (\overline{M}_{s,\tau-\epsilon}\rho_{\epsilon})(z), \quad \tau \in \mathbb{C}_+, z \in K_{\mathbb{C}}$$

is analytic in both variables.

We claim that for $\tau = t \in \mathbb{R}_+$ and $z = x \in K$, $\rho_{\mathbb{C}}(t, x) = \rho_t(x)$; once we establish this, the proof is complete by the uniqueness of analytic continuation. To prove this point, let f_n be a sequence of matrix entries on K with $\|f_n - \rho_{\epsilon}\|_{L^2(K, \rho_s)} \rightarrow 0$. From Definition 4.26 and Proposition 2.45, we have

$$(M_{s,t-\epsilon}f_n)(x) = \left(e^{\frac{t-\epsilon}{2}\Delta_{\mathfrak{t}}}f_n\right)(x) = \int_K f_n(y)\rho_{t-\epsilon}(y^{-1}x)dy. \quad (4.27)$$

On the other hand, Theorem 2.30(3) yields

$$\rho_t(x) = \int_K \rho_{\epsilon}(y)\rho_{t-\epsilon}(y^{-1}x)dy. \quad (4.28)$$

Combining (4.27) and (4.28), it then follows that, for each $x \in K$,

$$\begin{aligned} (M_{s,t-\epsilon}f_n)(x) - \rho_t(x) &= \int_K (f_n(y) - \rho_{\epsilon}(y))\rho_{t-\epsilon}(y^{-1}x)dy \\ &= \int_K (f_n(y) - \rho_{\epsilon}(y))\frac{\rho_{t-\epsilon}(y^{-1}x)}{\rho_s(y)}\rho_s(y)dy. \end{aligned}$$

Because K is compact, the functions $\rho_{t-\epsilon}$ and ρ_s are continuous and bounded strictly above 0. Hence the function $\omega_{s,t-\epsilon}(x, y) = \rho_{t-\epsilon}(y^{-1}x)/\rho_s(y)$ is continuous, and therefore in $L^2(K, \rho_s)$. Ergo, by the Cauchy–Schwarz inequality,

$$|(M_{s,t-\epsilon}f_n)(x) - \rho_t(x)| \leq \|f_n - \rho_{\epsilon}\|_{L^2(K, \rho_s)} \|\omega_{s,t-\epsilon}(x, \cdot)\|_{L^2(K, \rho_s)} \rightarrow 0 \quad (4.29)$$

for each $x \in K$. On the other hand, by the definition of $\overline{M}_{s,t-\epsilon}$, since $f_n \rightarrow \rho_{\epsilon}$ in $L^2(K, \rho_s)$, it follows that $M_{s,t-\epsilon}f_n \rightarrow \overline{M}_{s,t-\epsilon}\rho_{\epsilon} = \rho_{\mathbb{C}}(t, \cdot)$ in $\mathcal{H}L^2(K_{\mathbb{C}}, \mu_{s,\tau})$. The

presence of pointwise bounds in this reproducing kernel Hilbert space mean that we also have pointwise convergence of $M_{s,t-\epsilon}f_n \rightarrow \rho_{\mathbb{C}}(t, \cdot)$. Combining this with (4.29) shows that $\rho_{\mathbb{C}}(t, x) = \rho_t(x)$ for $t > 0$ and $x \in K$. Thus, $\rho_{\mathbb{C}}(\tau, z)$ is the analytic continuation of $\rho_t(x)$, as claimed. \square

Now having the analytic continuation $\rho_{\mathbb{C}}$ in hand, we can nominally define the Segal–Bargmann transform $B_{s,\tau}$ as in (1.5): for $f \in L^2(K, \rho_s)$,

$$(B_{s,\tau}f)(z) = \int_K f(x) \rho_{\mathbb{C}}(\tau, x^{-1}z) dx, \quad \text{for } \tau \in \mathbb{C}_+, z \in K_{\mathbb{C}}.$$

There is no question about convergence here: the function $x \mapsto \rho_{\mathbb{C}}(\tau, x^{-1}z) dx$ is continuous, hence in $L^2(K)$ since K is compact; therefore, since $f \in L^2(K, \rho_s) = L^2(K)$, the integral converges. It is also straightforward to show that this defines a holomorphic function.

Lemma 4.33. *The function $\mathbb{C}_+ \times K_{\mathbb{C}} \ni (\tau, z) \mapsto (B_{s,\tau}f)(z)$ is holomorphic.*

Proof. The function $(\tau, z) \mapsto f(x) \rho_{\mathbb{C}}(\tau, x^{-1}z)$ is holomorphic, and since the x -integral is over a compact space, standard techniques show that we can differentiate under the integral in τ and z . It follows from the Cauchy–Riemann equations, and the holomorphicity of the integrand, that the $(B_{s,\tau}f)(z)$ is holomorphic in both τ and z . \square

In fact, we now show that $B_{s,\tau} = \overline{M}_{s,\tau}$, which (thanks to Proposition 4.29) proves Theorem 1.3 in the compact case.

Proof of Theorem 1.3 in the compact group case. First, if f is a matrix entry, then for $\tau = t > 0$ and $z = x \in K$, we have

$$(B_{s,t}f)(x) = \int_K f(y) \rho_t(y^{-1}x) dy = \left(e^{\frac{t}{2}\Delta_{\mathfrak{t}}} f \right)(x) = (M_{s,t}f)(x)$$

by Theorem 2.45 and Definition 4.26. Now, both functions are holomorphic in τ and z , and therefore we must have $(B_{s,\tau}f)(z) = (M_{s,\tau}f)(z) = (\overline{M}_{s,\tau}f)(z)$ for $(\tau, z) \in \mathbb{C}_+ \times K_{\mathbb{C}}$.

So $B_{s,\tau}$ and $\overline{M}_{s,\tau}$ agree on the dense subspace of matrix elements. Now, let $f \in L^2(K, \rho_s)$, and let f_n be a sequence of matrix entries with $\|f_n - f\|_{L^2(K, \rho_s)} \rightarrow 0$. As above, let

$$\omega_{s,\tau}(x, z) = \frac{\rho_{\mathbb{C}}(\tau, x^{-1}z)}{\rho_s(x)}.$$

This function is continuous in x , ergo in $L^2(K, \rho_s)$. Then for any $z \in K_{\mathbb{C}}$, we have

$$\begin{aligned} |(M_{s,\tau}f_n)(z) - (B_{s,\tau}f)(z)| &= |(B_{s,\tau}f_n)(z) - (B_{s,\tau}f)(z)| \\ &= \left| \int_K (f_n(x) - f(x)) \rho_{\mathbb{C}}(\tau, x^{-1}z) dx \right| \\ &= \left| \int_K (f_n(x) - f(x)) \omega_{s,\tau}(x, z) \rho_s(x) dx \right| \\ &\leq \|f_n - f\|_{L^2(K, \rho_s)} \|\omega_{s,\tau}(\cdot, z)\|_{L^2(K, \rho_s)} \rightarrow 0. \end{aligned}$$

Hence, $B_{s,\tau}f$ is the pointwise limit of $M_{s,\tau}f_n$. As $\overline{M}_{s,\tau}f$ is the $\mathcal{H}L^2(K_{\mathbb{C}}, \mu_{s,\tau})$ -limit, and therefore the pointwise limit (by Proposition 4.28), it follows that $B_{s,\tau}f = \overline{M}_{s,\tau}f$, concluding the proof. \square

4.2.4. *The Proof of Theorem 1.4.* Here we are concerned with the action of the transform on functions in $L^2(K)$; since K is compact, $L^2(K) = L^2(K, \rho_s)$ as vector spaces, and so all of the discussion in the preceding section applies equally well in this context. As usual, we begin with the action on matrix entries. As the range measure ν_t involves averaging over K (cf. (3.14)), it is convenient to note the following.

Lemma 4.34. *Let f be a matrix entry on K or $K_{\mathbb{C}}$. Then the right- K -average*

$$\hat{f}(x) = \int_K f(xy) dy$$

is also a matrix entry.

Proof. Let $f = f_{\pi, A}$. We simply compute that

$$\hat{f}(x) = \int_K f_{\pi, A}(xy) dy = \int_K \text{Tr}(\pi(xy)A) dy = \int_K \text{Tr}(\pi(x)\pi(y)A) dy = \text{Tr}(\pi(x)\hat{A}_{\pi})$$

where $\hat{A}_{\pi} = \int_K \pi(y)A dy$. Thus $\hat{f} = f_{\pi, \hat{A}_{\pi}}$. \square

Corollary 4.35. *Fix an orthonormal basis $\{X_j\}_{j=1}^d$ for \mathfrak{k} , let $Y_j = JX_j$, and denote $\Delta_{i\mathfrak{k}} = \sum_{j=1}^d \tilde{Y}_j^2$. If f is a matrix entry on $K_{\mathbb{C}}$, and \hat{f} is its right- K -average, then for $t > 0$,*

$$\int_{K_{\mathbb{C}}} f(z) \nu_t(z) dz = \left(e^{\frac{t}{4}\Delta_{i\mathfrak{k}}} \hat{f} \right) (e) = \sum_{n=0}^{\infty} \frac{(t/4)^n}{n!} \left((\Delta_{i\mathfrak{k}})^n \hat{f} \right) (e).$$

Proof. We follow the proof of Lemma 3.8. Applying Fubini's theorem (justified by the exponential growth bounds of f), we compute that

$$\int_{K_{\mathbb{C}}} f(z) \nu_t(z) dz = \int_{K_{\mathbb{C}}} \hat{f}(z) \mu_{s, \tau}(z) dz = \left(e^{\frac{1}{2}\Delta_{s, \tau}} \hat{f} \right) (e) \quad (4.30)$$

where the last equality is justified by Proposition 2.45 and the fact that \hat{f} is a matrix entry by Lemma 4.34. Precisely as in the proof of Lemma 3.8, the right- K -invariance of \hat{f} implies that $\tilde{X}\hat{f} = 0$ for any $X \in \mathfrak{k}$, and so

$$\Delta_{s, \tau} \hat{f} = \sum_{j=1}^d \left[\left(s - \frac{t}{2} \right) \tilde{X}_j^2 + \frac{t}{2} \tilde{Y}_j^2 - u \tilde{Y}_j \tilde{X}_j \right] \hat{f} = \frac{t}{2} \sum_{j=1}^d \tilde{Y}_j^2 \hat{f} = \frac{t}{2} \Delta_{i\mathfrak{k}} \hat{f}.$$

Combining this with (4.30) completes the proof. \square

Proposition 4.36. *Let f be a matrix entry on K . Let $\tau \in \mathbb{C}_+$ with $\text{Re } \tau = t > 0$. Then*

$$\|B_{\infty, \tau} f\|_{L^2(K_{\mathbb{C}}, \nu_t)} = \|f\|_{L^2(K)}.$$

Proof. The action of $B_{\infty, \tau}$ (which is independent of the $s = \infty$ variable) on matrix elements is given (according to the proof of Theorem 1.3 in Section 4.2.3) by Definition 4.26:

$$B_{\infty, \tau} f = e^{\frac{\tau}{2}\Delta_{i\mathfrak{k}}} f_{\mathbb{C}}.$$

Following the proof of Theorem 4.27, if $F = B_{\infty, \tau} f$ then

$$|F|^2 = e^{(\frac{\tau}{2}\partial^2 + \frac{\bar{\tau}}{2}\bar{\partial}^2)} |f_{\mathbb{C}}|^2. \quad (4.31)$$

Since $|F|^2$ is a matrix entry (cf. Lemma 2.38), we may apply Corollary 4.35 to compute that

$$\int_{K_{\mathbb{C}}} |F(z)|^2 \nu_t(z) dz = \left(e^{\frac{t}{4}\Delta_{i\mathfrak{k}}} \widehat{|F|^2} \right) (e). \quad (4.32)$$

By Lemma 4.22, both ∂^2 and $\bar{\partial}^2$ commute with the right action of K , and hence

$$\widehat{|F|^2} = e^{(\frac{\tau}{2}\partial^2 + \frac{\bar{\tau}}{2}\bar{\partial}^2)} \widehat{|f_{\mathbb{C}}|^2}. \quad (4.33)$$

Combining (4.31), (4.32), and (4.33) yields

$$\|F\|_{L^2(K_{\mathbb{C}}, \nu_t)}^2 = \left(e^{\frac{t}{4}\Delta_{i\mathfrak{k}} + \frac{\tau}{2}\partial^2 + \frac{\bar{\tau}}{2}\bar{\partial}^2} \widehat{|f_{\mathbb{C}}|^2} \right) (e) \quad (4.34)$$

where we have used the fact that ∂^2 and $\bar{\partial}^2$ commute with $\Delta_{i\mathfrak{k}}$ (cf. Proposition 4.23). A simple computation using (4.22) and (4.23) shows that, if $\text{Im } \tau = u$, then

$$\frac{t}{4}\Delta_{i\mathfrak{k}} + \left(\frac{\tau}{2}\partial^2 + \frac{\bar{\tau}}{2}\bar{\partial}^2 \right) = \frac{t}{4}\Delta_{i\mathfrak{k}} + i\frac{u}{2}(\partial^2 - \bar{\partial}^2) \quad (4.35)$$

and the three operators in this sum commute by Corollary 4.24. Moreover, from (4.23) we may write the operator as

$$\frac{t}{4}\Delta_{i\mathfrak{k}} - \frac{u}{2} \sum_{j=1}^d \tilde{X}_j \tilde{Y}_j = \sum_{j=1}^d \left(\tilde{X}_j - \frac{u}{2} \tilde{Y}_j \right) \tilde{X}_j \quad (4.36)$$

where we have commuted X_j and Y_j , cf. Corollary 2.3.

Finally, since $\hat{h} = \widehat{|f_{\mathbb{C}}|^2}$ is K -right-invariant, $\tilde{X}\hat{h} = 0$ for all $X \in \mathfrak{k}$. Thus (4.36) kills \hat{h} , and so the exponential of (4.36) fixed \hat{h} . Ergo, (4.34) yields

$$\|F\|_{L^2(K_{\mathbb{C}}, \nu_t)}^2 = \widehat{|f_{\mathbb{C}}|^2}(e) = \int_K |f_{\mathbb{C}}(ex)|^2 dx = \int_K |f(x)|^2 dx = \|f\|_{L^2(K)}^2$$

and this concludes the proof. \square

Remark 4.37. Here is the outline of a different proof, appealing to the comparable theorem in the real time case proved in [16, Theorem 2]. We decompose $\tau\Delta_{i\mathfrak{k}} = t\Delta_{i\mathfrak{k}} + iu\Delta_{i\mathfrak{k}}$ to write

$$B_{\infty, \tau} f = \left(e^{\frac{t}{2}\Delta_{i\mathfrak{k}}} e^{i\frac{u}{2}\Delta_{i\mathfrak{k}}} f \right)_{\mathbb{C}} = B_{\infty, t} f^u$$

where

$$f^u = e^{i\frac{u}{2}\Delta_{i\mathfrak{k}}} f.$$

The operator $e^{i\frac{u}{2}\Delta_{i\mathfrak{k}}}$ is unitary, and so $\|f^u\|_{L^2(K)} = \|f\|_{L^2(K)}$. Hence, to prove the proposition, it suffices to show that $\|B_{\infty, t} f\|_{L^2(\nu_t)} = \|f\|_{L^2(K)}$ for all matrix entries f ; this is precisely the statement of [16, Theorem 2].

The remainder of the proof of Theorem 1.4 now follows precisely the same outline as the proofs of Theorem 1.3 in Section 4.1.4: we extend the transform from the dense subspace of matrix entries to an isometry from $L^2(K)$ into $\mathcal{H}L^2(K_{\mathbb{C}}, \nu_t)$ by continuity; since every matrix entry is in the range of the transform (precisely as in the proof of Lemma 4.13), this extension is also a surjection (owing to the density of holomorphic matrix entries in $\mathcal{H}L^2(K_{\mathbb{C}}, \nu_t)$, which was proved in Theorem 3.11), hence it is an isometric isomorphism. We then identify this extension as the original transform as above, using the pointwise bounds in the reproducing kernel Hilbert space $\mathcal{H}L^2(K_{\mathbb{C}}, \nu_t)$ to convert the L^2 -limit defining the extension to a pointwise limit. The details are left to the reader.

4.3. The General Compact Type Group Case. Let H be a compact type Lie group, with Lie algebra \mathfrak{h} . Proposition 2.10 asserts that there is a compact Lie group K (with Lie algebra \mathfrak{k}) and some $k \in \mathbb{N}$ so that H is isometrically isomorphic to $K \times \mathbb{R}^k$. In particular, this means $\mathfrak{h} = \mathfrak{k} \oplus \mathbb{R}^k$, and the $\text{Ad}(H)$ -invariant inner product has the form $\langle \cdot, \cdot \rangle_{\mathfrak{k}} + \langle \cdot, \cdot \rangle_{\mathbb{R}^k}$ for some $\text{Ad}(K)$ -invariant inner product on \mathfrak{k} , and the standard inner product on \mathbb{R}^k .

Let us denote elements of H as (x, y) , with $x \in K$ and $y \in \mathbb{R}^k$. Since the subgroups K and \mathbb{R}^k commute, it is then routine to check that the heat kernel $\rho_t^{\Delta_{\mathfrak{h}}}(x, y)$ has the form

$$\rho_t^{\Delta_{\mathfrak{h}}}(x, y) = \rho_t^{\Delta_{\mathfrak{k}}}(x) \rho_t^{\Delta_{\mathbb{R}^k}}(y). \quad (4.37)$$

Now, the complexification of H is $H_{\mathbb{C}} = K_{\mathbb{C}} \times \mathbb{C}^k$; denote points in it by (z, w) . Theorem 1.1, thus far proved for K and \mathbb{R}^k separately, thus allows us to define

$$\rho_{\mathbb{C}}^{H_{\mathbb{C}}}(\tau, (z, w)) = \rho_{\mathbb{C}}^{K_{\mathbb{C}}}(\tau, z) \rho_{\mathbb{C}}^{\mathbb{C}^k}(\tau, w) \quad (4.38)$$

which is holomorphic on $H_{\mathbb{C}}$ and restricts to the heat kernel (4.37) on H at real time, thus proving Theorem 1.1 in general.

The product structure of the complexified heat kernel shows that if $f \in L^2(H, \rho_s^H)$ happens to be a product function $f(x, y) = f_1(x)f_2(y)$ with $f_1 \in L^2(K, \rho_s^K)$ and $f_2 \in L^2(\mathbb{R}^k, \rho_s^{\mathbb{R}^k})$, then the formula (1.5) defining the transform for the compact and Euclidean cases separately yields

$$(B_{s,\tau}^H f)(z, w) = (B_{s,\tau}^K f_1)(z) (B_{s,\tau}^{\mathbb{R}^k} f_2)(w)$$

and this function is certainly holomorphic on the product space. Since the complexification is also a product, we find that, as with the heat kernel on K ,

$$\mu_{s,\tau}^H(z, w) = \mu_{s,\tau}^K(z) \mu_{s,\tau}^{\mathbb{C}^k}(w).$$

The isometry Theorem 1.3 for the two separate cases, together with Tonelli's theorem, shows that

$$\begin{aligned} \|B_{s,\tau}^H f\|_{L^2(H, \mu_{s,\tau}^H)}^2 &= \|B_{s,\tau}^K f_1\|_{L^2(K, \mu_{s,\tau}^K)}^2 \|B_{s,\tau}^{\mathbb{R}^k} f_2\|_{L^2(\mathbb{R}^k, \mu_{s,\tau}^{\mathbb{R}^k})}^2 \\ &= \|f_1\|_{L^2(K, \rho_s^K)}^2 \|f_2\|_{L^2(\mathbb{R}^k, \rho_s^{\mathbb{R}^k})}^2 \\ &= \|f\|_{L^2(H, \rho_s^H)}^2. \end{aligned}$$

Hence, $B_{s,\tau}^H$ is an isometry when acting on elements in $L^2(H, \rho_s^H) \cong L^2(K, \rho_s^K) \otimes L^2(\mathbb{R}^k, \rho_s^{\mathbb{R}^k})$ of the form $f_1 \otimes f_2$. Applying this to the standard orthonormal tensor basis shows that $B_{s,\tau}^H$ is an isometry from $L^2(H, \rho_s^H)$ into $\mathcal{H}L^2(H_{\mathbb{C}}, \mu_{s,\tau}^H)$. Additionally, the codomain is isomorphic to $\mathcal{H}L^2(K_{\mathbb{C}}, \mu_{s,\tau}^K) \otimes \mathcal{H}L^2(\mathbb{C}^k, \mu_{s,\tau}^{\mathbb{C}^k})$, and since linear combinations of elements of the form $h_1 \otimes h_2$ are dense in this space, the surjectivity of the transform in the two separate cases implies the surjectivity of the product transform. This completes the proof of Theorem 1.3 in the general case.

APPENDIX A. ESSENTIAL SELF-ADJOINTNESS OF THE LAPLACIAN

This section provides a self-contained proof that, on any Lie group, any “sum of squares” Laplacian is essentially self-adjoint, with $C_c^\infty(G)$ as a core. This proof is adapted from notes due to L. Gross.

Let G be a real Lie group with Lie algebra \mathfrak{g} , on which we fix an inner product throughout. Let $\{X_j\}_{j=1}^k$ be a collection of left invariant vector fields on G , and define

$$L_0 := \sum_{j=1}^k X_j^2$$

acting on $C^2(G)$ and let $L := L_0|_{C_c^\infty(G)}$, i.e. $L = L_0$ on $\mathcal{D}(L) := C_c^\infty(G)$. Further let λ denote a right invariant Haar measure on G .

Theorem A.1. *The second order differential operator L is essentially self-adjoint on $L^2(G, \lambda)$, with $C_c^\infty(G)$ as a core.*

Before giving the proof we will need a little notation and a few preparatory results.

Notation A.2. For $u \in C_c^\infty(G)$ and f and $g \in L^2(G)$, define the **convolution** of u and f by

$$(u * f)(x) = \int_G u(xy^{-1})f(y) \lambda(dy). \quad (\text{A.1})$$

We will also let

$$\tilde{u}(x) = \bar{u}(x^{-1})$$

which should not be confused with the notation $\tilde{\xi}$ for the left invariant vector field determined by an element $\xi \in \mathfrak{g}$.

Proposition A.3. Assume $f, g \in L^2(G, \lambda)$.

(1) If $u \in C_c^\infty(G)$, then $u * f \in L^2$ and

$$\|u * f\|_2 \leq \left(\int_G |u| \sqrt{m} d\lambda \right) \|f\|_2$$

where m is the modular function of G .

(2) $\langle u * f, g \rangle_{L^2(G, \lambda)} = \langle f, \tilde{u} * g \rangle_{L^2(G, \lambda)}$ for all $u \in C_c^\infty(G)$.

(3) For $X \in \mathfrak{g}$, $\tilde{X}(u * v) = u * (\tilde{X}v)$ for all $u \in C_c^\infty(G)$ and $v \in C^\infty(G)$.

(4) $\langle \tilde{X}u, v \rangle_{L^2(G, \lambda)} = -\langle u, \tilde{X}v \rangle_{L^2(G, \lambda)}$ for all $u \in C^\infty(G)$ and $v \in C_c^\infty(G)$.

(5) There exist $u_n \in C_c^\infty(G, \mathbb{R})$ such that $u_n * \rightarrow I$ strongly on $L^2(G, \lambda)$.

Proof. In the following argument, we will use the right invariance of Haar measure, the definition of convolution in (A.1), and the left invariance of \tilde{X} without further mention.

Using the definition of the modular function, $\|g \circ L_x\|_2^2 = m(x) \|g\|_2^2$ for all $x \in G$. Therefore,

$$\begin{aligned} \int_{G^2} |u(xy^{-1})| |f(y)| |g(x)| \lambda(dx) \lambda(dy) &= \int_{G^2} |u(x)| |f(y)| |g(xy)| \lambda(dx) \lambda(dy) \\ &\leq \int_G |u(x)| \|f\|_2 \cdot \|g \circ L_x\|_2 \lambda(dx) \\ &\leq \int_G |u(x)| \sqrt{m(x)} \lambda(dx) \cdot \|f\|_2 \cdot \|g\|_2. \end{aligned}$$

This proves item (1) as a consequence of the converse to Hölder's inequality. It also justifies the use of Fubini's theorem used to prove item (2):

$$\begin{aligned} \langle u * f, g \rangle_{L^2(G, \lambda)} &= \int_{G^2} u(xy^{-1}) f(y) \bar{g}(x) \lambda(dx) \lambda(dy) \\ &= \int_{G^2} f(y) \overline{u(xy^{-1}) g(x)} \lambda(dx) \lambda(dy) = \langle f, \tilde{u} * g \rangle_{L^2(G, \lambda)}. \end{aligned}$$

For items (3) and (4), we have

$$\begin{aligned}\tilde{X}(u * v)(x) &= \frac{d}{dt} \Big|_{t=0} \int_G u(xe^{tX}y^{-1})v(y)\lambda(dy) \\ &= \frac{d}{dt} \Big|_{t=0} \int_G u(xy^{-1})v(ye^{tX})\lambda(dy) = u * \tilde{X}v\end{aligned}$$

and

$$\begin{aligned}\langle Xu, v \rangle_{L^2(G, \lambda)} &= \frac{d}{dt} \Big|_{t=0} \int_G u(xe^{tX})\bar{v}(x)\lambda(dx) \\ &= \frac{d}{dt} \Big|_{t=0} \int_G u(x)\bar{v}(xe^{-tX})\lambda(dx) = -\langle u, \tilde{X}v \rangle_{L^2(G, \lambda)}.\end{aligned}$$

For item (5) we apply the usual approximate identity sequence arguments to any sequence of functions $\{u_n\}_{n=1}^\infty \subset C_c^\infty(G, [0, \infty))$ with the following properties: 1) $\int_G u_n d\lambda = 1$ for all n and 2) $\text{supp}(u_n) \downarrow \{e\}$ as $n \rightarrow \infty$. \square

Lemma A.4. *For $f \in C^\infty(G) \cap \mathcal{D}(L^*)$, $L^*f = L_0f$ and moreover $C^\infty(G) \cap \mathcal{D}(L^*)$ is core for L^* .*

Proof. If $f \in C^\infty(G) \cap \mathcal{D}(L^*)$ and $v \in C_c^\infty(G)$, then by the definition of L^* and repeated use of Proposition A.3,

$$\langle L^*f, v \rangle_{L^2(G, \lambda)} = \langle f, Lv \rangle_{L^2(G, \lambda)} = \langle L_0f, v \rangle_{L^2(G, \lambda)}.$$

Since $v \in C_c^\infty(G)$ is arbitrary, it follows that $L^*f = L_0f$.

Now suppose that $f \in \mathcal{D}(L^*)$ and that $u, v \in C_c^\infty(G)$. Then $\tilde{u} * v \in C_c^\infty(G)$ and therefore

$$\begin{aligned}\langle u * L^*f, v \rangle_{L^2(G, \lambda)} &= \langle L^*f, \tilde{u} * v \rangle_{L^2(G, \lambda)} = \langle f, L(\tilde{u} * v) \rangle_{L^2(G, \lambda)} \\ &= \langle f, \tilde{u} * Lv \rangle_{L^2(G, \lambda)} = \langle u * f, Lv \rangle_{L^2(G, \lambda)}.\end{aligned}$$

It follows from this equation that $u * f \in \mathcal{D}(L^*)$ and that

$$L^*(u * f) = u * L^*f \text{ for all } u \in C_c^\infty(G). \quad (\text{A.2})$$

Now choose u_n as in Proposition A.3(5). Since each $u_n * f \in C^\infty(G) \cap \mathcal{D}(L^*)$, the lemma follows from (A.2). \square

Notation A.5. *The tensor $D^n f(x) \in (\mathfrak{g}^{\otimes n})^*$ of n^{th} -order derivatives of f at x is defined by*

$$\langle (D^n f)(x), \xi_1 \otimes \cdots \otimes \xi_n \rangle = \left(\tilde{\xi}_1 \cdots \tilde{\xi}_n f \right)(x) \quad (\text{A.3})$$

where $\xi_j \in \mathfrak{g}$ for $1 \leq j \leq n$, and the inner product is the standard one induced by the inner product on \mathfrak{g} .

Let us now recall from [6, Lemma 3.6] that there exists $h_n \in C_c^\infty(G, [0, 1])$ such that h_n is increasing, $h_n^{-1}(\{1\}) \uparrow G$ as $n \uparrow \infty$, and $\sup_n \sup_{x \in G} |D^k h_n(x)| < \infty$ for any $k \in \mathbb{N}$ where $D^k h_n$ is as defined in (A.3).

Lemma A.6. *If $f \in C^\infty(G, \mathbb{R}) \cap \mathcal{D}(L^*)$, then*

$$\int_G \sum_{j=1}^k (\tilde{X}_j f)^2 d\lambda = -\langle L^*f, f \rangle_{L^2(G, \lambda)} < \infty \quad (\text{A.4})$$

Proof. For $h \in C_c^\infty(G, [0, \infty))$,

$$\begin{aligned} \sum_{j=1}^k \int_G h(x) (\tilde{X}_j f)^2 \lambda(dx) &= - \sum_j \int_G h(x) (\tilde{X}_j^2 f) f + (\tilde{X}_j h) (\tilde{X}_j f) f d\lambda \\ &= - \int_G h(x) (L^* f) f d\lambda - \frac{1}{2} \sum_j \int_G (\tilde{X}_j h) \tilde{X}_j (f^2) d\lambda \\ &= - \langle h L^* f, f \rangle_{L^2(G, \lambda)} + \frac{1}{2} \int_G (Lh) f^2 d\lambda. \end{aligned} \quad (\text{A.5})$$

Now replace h in the above identity by h_n as in [6, Lemma 3.6], so in particular $h_n \uparrow 1$ as $n \rightarrow \infty$ and $Lh_n \rightarrow 0$ boundedly. Use the monotone convergence theorem on the left of (A.5) and the dominated convergence theorem on both terms on the right to verifies the truth of (A.4). \square

Proof of Theorem A.1. Let \overline{L} denote the closure of L . By Proposition A.3, each X_j is skew symmetric on $C_c^\infty(G)$ and as a consequence L is symmetric on $C_c^\infty(G)$. That is $L \subset L^*$ and therefore $\overline{L} \subset L^* = \overline{L}^*$. So it remains only to show $L^* \subset \overline{L}$, or equivalently that $C_c^\infty(G)$ is a core for L^* .

Using Lemma A.4, it suffices to prove the following: for every $f \in C^\infty(G) \cap \mathcal{D}(L^*)$, there exists $f_n \in C_c^\infty(G)$ such that f_n converges to f in the L^* -graph norm. Choose $0 \leq h_n \leq 1$ with $h_n \in C^\infty(G)$ as in [6, Lemma 3.6] and let $f_n(x) = h_n(x)f(x)$. Then $f_n \in C^\infty(G)$ and $f_n \rightarrow f$ in L^2 since $h_n \uparrow 1$. Moreover,

$$L f_n = (L h_n) f + h_n (L^* f) + 2 \sum_{j=1}^k (X_j h_n) (X_j f) \quad (\text{A.6})$$

and $L h_n \rightarrow 0$ boundedly by [6, Lemma 3.6]. The first two terms therefore together converge to $L^* f$. Since $X_j h_n \rightarrow 0$ pointwise and boundedly, Lemma A.6 implies the third term in Eq. (A.6) converges to zero in $L^2(G, \lambda)$. Thus $L f_n \rightarrow L^* f$ in $L^2(G, \lambda)$, concluding the proof. \square

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